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The nonlinear evolution of inviscid Görtler vortices in three-dimensional boundary layers

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Contents

The nonlinear development of inviscid Görtler vortices in a three-dimensional boundary-layer flow is considered by extending the theory of Blackaby *et al.*, who consider the closely related problem concerning the nonlinear development of disturbances in stratified shear flows. The inviscid Görtler modes considered are initially unstable (and hence growing) based on linear theory; however, following others, we assume that the effects of boundary-layer spreading result in them evolving in a linear fashion until they reach a stage where their amplitudes are sufficiently large, and their growth rates have diminished sufficiently, such that their subsequent evolution can be considered within the framework of a weakly nonlinear theory based on a nonlinear, non-equilibrium critical-layer theory. As with the closely related stratified-shear flow problem, three possible nonlinear integro-differential evolution equations for the disturbance amplitude should arise; however, it is found that only two of these are in fact possible. One of the possible integro-differential evolution equations has a cubic-nonlinearity due to supercriticality (non-neutrality) effects, while the other amplitude evolution equation has a quintic nonlinearity but is only relevant for larger sizes of disturbance. Thus, in this paper, attention is concentrated on the former, since this equation is appropriate earlier in the evolution process of the Görtler modes. Numerical results are presented which demonstrate that variations in the level of cross-flow present in the underlying flow have a significant effect on the nonlinear problem, as they do on the linear problem. It is found that the consideration of a spatial evolution problem (as opposed to a temporal stability approach adopted in the above paper) leads to significant changes to the resulting evolution equations.

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1. Introduction

Since the work of Görtler (1940) there has been considerable interest in the boundary layer instability mechanism named after him. Much of the early work on Görtler vortices was shown to be flawed, by Hall (1982*a*, *b*), because it invoked the parallel flow approximation and thus ignored the effects of boundary-layer growth. Hall (1983) went on to show that for Görtler vortices of order-one wavenumber the ideas of unique neutral curves and growth rates are untenable. The stability properties of such modes depend upon the initial form and location of the disturbance.

Since the early 1980s there have been numerous theoretical studies on Görtler vortices; the reader is referred to the review papers by Hall (1990) and Saric (1994) for an overview of the subject. Most of these papers have concentrated on two-dimensional boundary layers but obviously many practical situations in which Görtler vortices arise will be three-dimensional in nature. One example of particular interest has been caused by the development of laminar flow control airfoils which have areas of concave curvature on the underside of the airfoil near the leading edge. When the wing is swept the flow becomes fully three-dimensional. Consequently several recent studies have considered the stability of Görtler vortices in three-dimensional boundary layers and this three-dimensionality has been shown to have an important effect upon the stability properties of these vortices.

Bassom & Hall (1991) looked at the viscous and inviscid stability problems for an incompressible boundary layer flow, which could support both Görtler and crossflow vortices, over an infinitely long swept cylinder. They found that, for sufficiently large values of the parameter representing the degree of three-dimensionality of the flow, there are no Görtler vortices present in a boundary layer which, in the zero crossflow case, is centrifugally unstable. The inviscid stability problem has been extended to compressible boundary layers by Dando (1992). Similar results to the incompressible case are found; three-dimensionality has a stabilizing effect on vortices of all wavelengths except for a band of small wavelengths where the vortices are dominated by crossflow effects and are in fact of the type considered by Gregory *et al.* (1955). The numerical results of Dando (1992) showed that for larger Mach numbers a larger crossflow was needed to completely stabilize Görtler vortices over a band of wavenumbers and this was confirmed by the asymptotic study of Fu & Hall (1994), who considered the hypersonic limit.

One of the most interesting points to emerge from the work of Bassom & Hall (1991) and Dando (1992) was that in the presence of a relatively weak crossflow, Görtler vortex disturbances of all wavelengths are stabilized such that the inviscid modes possess some of the largest growth rates while also being neutral at certain other wavenumbers. Furthermore their governing equation has many similarities to the Taylor–Goldstein equation which governs the linear stability of stratified shear flows (see Goldstein 1931; Taylor 1931). In fact Blackaby & Choudhari (1993) have illustrated the close connection between the two problems of inviscid Görtler modes in three-dimensional boundary layers and modes on unstable stratified shear layers, and proposed a definition of a generalized Richardson number (this is the parameter which characterizes the stratification of a shear flow) for such centrifugally driven instabilities. The study of Bassom & Otto (1993) used a classical weakly nonlinear approach to consider the stability of $O(G^{1/5})$ wavenumber (here G is the Görtler number), essentially viscous, modes in three-dimensional boundary layers. In the present paper we restrict our attention to the $O(1)$ wavenumber inviscid modes.

It was this close connection between the two problems which encouraged the authors, in their desire to develop a theory describing the nonlinear evolution of these inviscid Görtler modes, to initially consider the nonlinear evolution of modes on stratified shear layers in Blackaby *et al.* (1993). In order to place the latter paper and the current study in context, and to understand the theory underpinning both, it is necessary to review some of the recent contributions to nonlinear critical-layer theory; a more general review of the theory can be found in the papers by Stewartson (1981) and Maslowe (1986). Over the last couple of decades much attention has been focused on the nonlinear stability of non-stratified shear layers. In the case where there is no vertical density variation the linear stability of the flow is usually governed by the familiar Rayleigh equation (to which the Taylor–Goldstein equation reduces for zero Richardson number). Benney & Bergeron (1969) developed the so-called equilibrium critical layer theory: here the mode is treated as 'quasi-steady' inside the critical layer as well as outside it. Nonlinearity affects the jump imposed across the critical layer and hence leads to modified results for the neutral (equilibrium) modes. Haberman (1972) extended the theory to include critical layers where viscosity is also significant. Some of the early studies of non-equilibrium critical layers include the work of Brown & Stewartson (1978), Warn & Warn (1978) and Hickernell (1984). The key paper by Hickernell (1984) concerned a shear layer affected by Coriolis (rotational) effects; here the weakly nonlinear theory leads to an integro-differential equation rather than the (previously) more familiar Stuart–Watson–Landau equation with its 'polynomial' nonlinear terms. In fact such integro-differential equations result naturally from non-equilibrium nonlinear critical layer theories when the shear layer is coupled with other physical factors such as, for instance, Coriolis effects (Hickernell 1984; Shukhman 1991); compressibility effects (Goldstein & Leib 1989); three-dimensionality effects (Goldstein & Choi 1989; Wu *et al.* 1993) and buoyancy effects (Churilov & Shukhman 1988; Blackaby *et al.* 1993). However, the case of a 'simple' shear layer, not affected by any additional physical factors, is a special case in the sense that it does not lead to an integro-differential equation; instead, Goldstein & Leib (1988) found that the nonlinear evolution of a disturbance was governed by the full unsteady nonlinear critical-layer equations.

This difference is due to the additional physical factors, of the former cases, resulting in stronger singularities of the inviscid disturbance quantities at the critical level.

At first sight, it appears that weakly nonlinear theories can only be usefully applied to marginally unstable flows; they rely on small growth rates and so the unstable disturbance of concern must be near to a neutral state. Thus it was believed that such theories are incapable of describing the initial evolution of 'far-from-neutral' unstable modes. However, several recent studies have derived integro-differential equations, using weakly nonlinear theories, to describe the nonlinear evolution of (general) unstable modes on a variety of shear layers (see the previous paragraph). These studies are based on the idea that, in actual physical flow situations, shear layer spreading or other external changes would result in the otherwise relatively unstable modes having their growth rates diminished in real terms, so that a weakly nonlinear critical-layer theory becomes appropriate. The work in this paper is based on the assumption that boundary layer growth acts in a similar manner to shear layer spreading. This theory is supported by the work of Michalke (1964), Crighton & Gaster (1976) and the excellent comparison with experiments recently achieved by Hultgren (1992). For further discussion of non-equilibrium critical layer theory the reader is directed to the reviews of Cowley & Wu (1993) and Goldstein (1994).

In this study we use weakly nonlinear and non-equilibrium critical-layer theories to describe the spatial, nonlinear development of inviscid, unstable Görtler modes in an incompressible, weakly three-dimensional boundary-layer. The theory of this paper is extendable to compressible boundary layers and also has obvious applications to inviscid modes in a flow above a heated plate, similar to those considered by Hall & Morris (1992).

While different from the approach adopted in this study, there are alternate or complementary nonlinear theories that have been developed recently in which two or more of the flow disturbances mutually interact. Such theories generally require smaller disturbance amplitudes but may also need the disturbances to exist in specific configurations. These other theories are generally referred to as wave–wave and vortex–wave interactions. For a discussion of wave–wave interactions and resonant triads the reader is directed to the book by Craik (1985). The ideas behind resonant triads and non-equilibrium critical layers have been combined in works by Goldstein & Lee (1992) and Wu (1992) which both consider resonant triad interactions where the growth rates of the disturbance are controlled by nonlinear interactions inside critical layers. Strongly nonlinear vortex–wave interactions were first considered by Hall & Smith (1991) and their ideas are clarified and extended by Brown *et al.* (1993); Smith *et al.* (1993) and Brown & Smith (1996). In fact, there are mathematical connections between these different nonlinear theories; for example, Wu *et al.* (1993) in their non-equilibrium, nonlinear-critical-layer study showed that the viscous limit of their amplitude equation is the same as the amplitude equation obtained by Smith *et al.* (1993) in their investigation of a vortex–wave interaction.

The rest of this paper is laid out as follows. In the next section we present some background details of the flow concerning us in this paper, namely inviscid Görtler vortices in three-dimensional incompressible boundary layers. In § 3 the flow outside the critical layer is considered; while § 4 covers the analysis of the flow inside the critical layer, concentrating on the derivation of the integro-differential amplitude evolution equation with cubic nonlinearity due to supercriticality effects. In § 5 we briefly consider the amplitude equation with cubic nonlinearity due to viscous effects. In § 6 we examine some numerical solutions for the amplitude evolution equation with cubic nonlinearity due to supercriticality effects, before finally drawing some conclusions in § 7.

2. Inviscid Görtler vortices in three-dimensional boundary layers: linear theory

It is helpful to recap the scalings and arguments that lead to the governing equation for inviscid Görtler vortices in three-dimensional boundary layers. In this paper we shall consider an incompressible flow. A more detailed derivation of the governing equation can be found in the papers by Bassom & Hall (1991), Dando (1992) and Fu & Hall (1994), for incompressible, compressible and hypersonic boundary-layer flows respectively.

The boundary layer considered is that of a flow over the infinite cylinder $y^* = 0$, $-\infty < z^* < \infty$ so that the z^*-axis is a generator of the cylinder and y^* measures the distance normal to the surface. The x^*-coordinate measures distance along the curved surface, which has variable curvature $(1/m^*)K(x^*/l^*)$ where m^* and l^* are typical length scales in the normal and streamwise directions. Here, the superscript asterisk is used to denote dimensional quantities; we shall non-dimensionalize our problem in due course. The Reynolds number, Re, Görtler number, G, and curvature parameter, δ, are defined by

$$Re = u_\infty^* l^*/\kappa^*, \qquad G = 2Re^{1/2}\delta, \qquad \delta = l^*/m^*, \tag{2.1}$$

where u_∞^* is a typical flow velocity in the streamwise direction and κ^* is the kinematic viscosity of the fluid.

The Reynolds number is assumed to be large, while δ is sufficiently small so that as $\delta \to 0$ the parameter G is fixed and of order one. The basic three-dimensional boundary layer is taken to be of the form

$$\underline{u} = u_\infty^*(\overline{u}(x,Y), \quad Re^{-1/2}\overline{v}(x,Y), \quad Re^{-1/2}\overline{\lambda}(x)\overline{w}(x,Y))(1 + O(Re^{-1/2})), \tag{2.2}$$

with the non-dimensionalizations and scalings

$$x = x^*/l^*, \qquad Y = Re^{1/2}y^*/l^*, \tag{2.3}$$

where the parameter $\overline{\lambda}$ is a measure of the relative strength of the crossflow present. The basic state is perturbed by writing

$$\underline{u} = u_\infty^*(\overline{u} + \epsilon\tilde{U}(x,Y)E,\ Re^{-1/2}\overline{v} + \epsilon Re^{-1/2}\tilde{V}(x,Y)E,$$
$$Re^{-1/2}\overline{\lambda}\,\overline{w} + \epsilon Re^{-1/2}\tilde{W}(x,Y)E)(1 + O(Re^{-1/2})), \tag{2.4 a}$$

$$p = \overline{p}(x) + \epsilon\, Re^{-1}\tilde{P}(x,Y)E + O(Re^{-3/2}), \tag{2.4 b}$$

where ϵ is a small parameter characterizing the magnitude of the vortex mode and

$$E = \exp\{\mathrm{i}aZ\}, \qquad Z = Re^{1/2}z^*/l^*. \tag{2.5}$$

We now consider the inviscid limit ($G \gg 1$) of the Görtler problem by introducing a scaled spatial growth rate, β, and the scalings

$$[\tilde{U}(x_0,Y), \tilde{V}(x_0,Y), \tilde{W}(x_0,Y), \tilde{P}(x_0,Y)]$$
$$= [\tilde{\tilde{U}}(Y), G^{1/2}\tilde{\tilde{V}}(Y), G^{1/2}\tilde{\tilde{W}}(Y), G\tilde{\tilde{P}}(Y)]\exp\left\{\int G^{1/2}\beta(x)\,\mathrm{d}x\right\}, \tag{2.6 a}$$

$$\overline{\lambda}(x_0) = G^{1/2}\lambda(x_0); \tag{2.6 b}$$

these scales were obtained independently by Timoshin (1990) and Denier *et al.* (1991) for two-dimensional boundary-layer flows. Here $x = x_0$ is the local streamwise location under consideration. Upon substituting these expansions into the continuity and momentum equations and letting $G \to \infty$ we obtain

$$\beta\tilde{\tilde{U}} + \tilde{\tilde{V}}_Y + \mathrm{i}a\tilde{\tilde{W}} = 0, \tag{2.7 a}$$

$$(\beta\overline{u} + \mathrm{i}a\lambda\overline{w})\tilde{\tilde{U}} + \overline{u}_Y\tilde{\tilde{V}} = 0, \tag{2.7 b}$$

$$(\beta\overline{u} + \mathrm{i}a\lambda\overline{w})\tilde{\tilde{V}} + K\overline{u}\tilde{\tilde{U}} = -\tilde{\tilde{P}}_Y, \tag{2.7 c}$$

$$(\beta\overline{u} + \mathrm{i}a\lambda\overline{w})\tilde{\tilde{W}} + \lambda\overline{w}_Y\tilde{\tilde{V}} = -\mathrm{i}a\tilde{\tilde{P}}, \tag{2.7 d}$$

from which it can easily be shown that $\tilde{\tilde{V}}$ satisfies

$$\tilde{\tilde{V}}_{YY} - \left(a^2 + \frac{(\beta\overline{u}_{YY} + \mathrm{i}a\lambda\overline{w}_{YY})}{(\beta\overline{u} + \mathrm{i}a\lambda\overline{w})} - \frac{a^2K\overline{u}\overline{u}_Y}{(\beta\overline{u} + \mathrm{i}a\lambda\overline{w})^2}\right)\tilde{\tilde{V}} = 0, \tag{2.8}$$

subject to the boundary conditions $\tilde{\tilde{V}}(0) = 0$ and $\tilde{\tilde{V}} \to 0$ as $Y \to \infty$. This is the equation that controls the inviscid growth of Görtler vortices in an incompressible, weakly three-dimensional boundary layer and, as noted earlier, it closely resembles the Taylor–Goldstein equation.

We choose to consider a Falkner–Skan–Cooke profile as a realistic yet relatively simple base flow; note that we cannot consider a Blasius flow as Hall (1985) showed that when $\overline{u}$ and $\overline{w}$ are linearly related the whole problem of Görtler vortices in three-dimensional boundary layers becomes degenerate and can be reduced to the two-dimensional case. In fact the base flow is chosen to have the similarity-solution form

$$\overline{u} = x^m f_{\hat{Y}}(\hat{Y}), \qquad \overline{w} = g(\hat{Y}), \qquad \hat{Y} = (\tfrac{1}{2}(1+m))^{1/2}x^{(m-1)/2}Y, \tag{2.9}$$

where f and g satisfy

$$\left.\begin{aligned} &f_{\hat{Y}\hat{Y}\hat{Y}} + ff_{\hat{Y}\hat{Y}} + \frac{2m}{m+1}(1-f_{\hat{Y}}^2) = 0, \qquad f(0) = f_{\hat{Y}}(0) = 0, \\ &f_{\hat{Y}}(\infty) = 1, \qquad g_{\hat{Y}\hat{Y}} + fg_{\hat{Y}} = 0, \qquad g(0) = 0, \qquad g(\infty) = 1. \end{aligned}\right\} \tag{2.10}$$

The value $m = \frac{1}{3}$ was chosen for the numerical results presented in this paper. This choice for m corresponds to that chosen in the linear-stability studies of Bassom & Hall (1991) and Dando (1992), although it is important to note that there is a minor error in the base flow used in these papers. In this paper we choose to scale a, β and K by writing

$$(a, \beta, K) = (\tfrac{1}{2}(1+m))^{1/2}(x^{(m-1)/2}\hat{a}, x^{-(m+1)/2}\hat{\beta}, x^{-3(m+1)/2}\hat{K}), \tag{2.11}$$

so that equation (2.8) becomes

$$\tilde{\tilde{V}}_{\hat{Y}\hat{Y}}(\hat{Y}) - \left(\hat{a}^2 + \frac{(\hat{\beta}f_{\hat{Y}\hat{Y}\hat{Y}} + \mathrm{i}\hat{a}\lambda g_{\hat{Y}\hat{Y}})}{(\hat{\beta}f_{\hat{Y}} + \mathrm{i}\hat{a}\lambda g)} - \frac{\hat{K}\hat{a}^2 f_{\hat{Y}}f_{\hat{Y}\hat{Y}}}{(\hat{\beta}f_{\hat{Y}} + \mathrm{i}\hat{a}\lambda g)^2}\right)\tilde{\tilde{V}}(\hat{Y}) = 0. \tag{2.12}$$

In figures 1*a*–*c* we present solutions of (2.12), showing plots of the scaled growth rate, $\hat{\beta}$, against the scaled spanwise wavenumber, $\hat{a}$, which illustrate the effect that crossflow has on inviscid Görtler vortices. These results are all for $\hat{K} = 1$ and $m = \frac{1}{3}$; further details can be found in the papers by Bassom & Hall (1991) and Dando

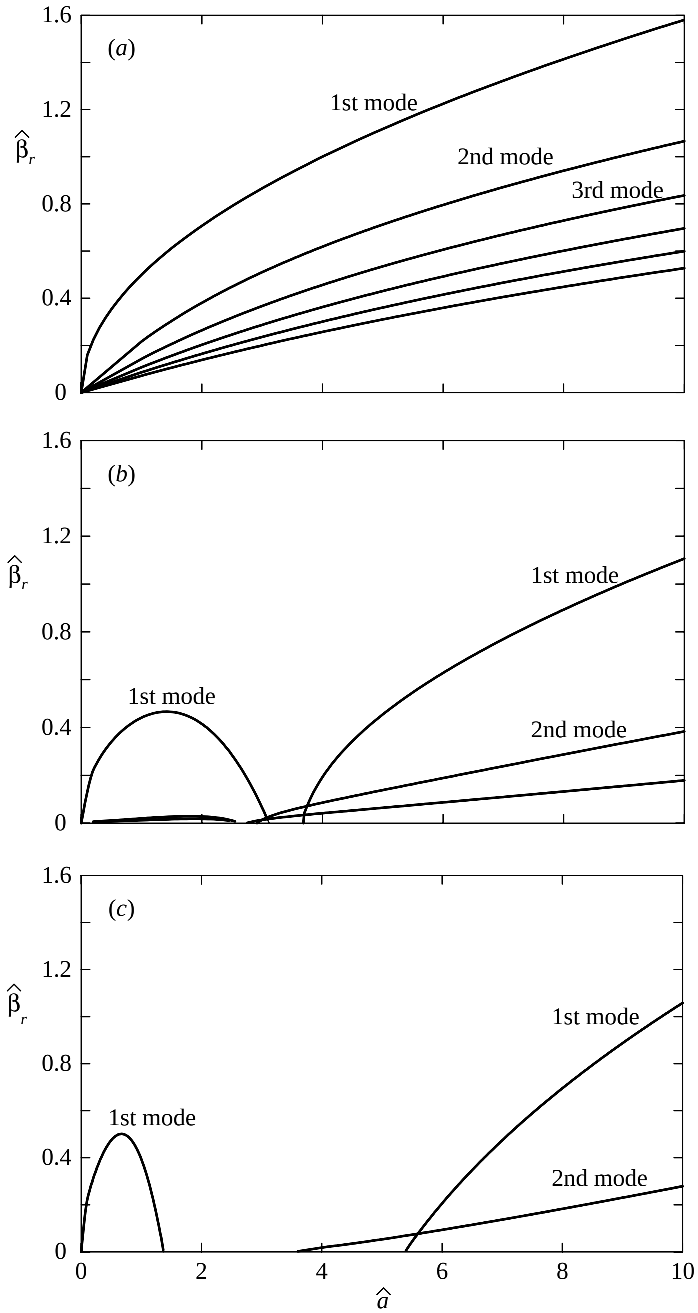

Figure 1. Real parts of the scaled spatial growth rate, $\hat{\beta}$, versus the scaled spanwise wavenumber, $\hat{a}$, for the cases when (a) there is no crossflow present (top figure), (b) $\lambda = 5$ (middle figure), and (c) $\lambda = 10$ (bottom figure).

(1992). Note that there are an infinite number of modes which, in the presence of a relatively weak crossflow, are stabilized for almost all wavelengths, such that the inviscid modes possess some of the largest growth rates while also attaining a neutral state at certain other wavenumbers. In their key study, Bassom & Hall (1991)

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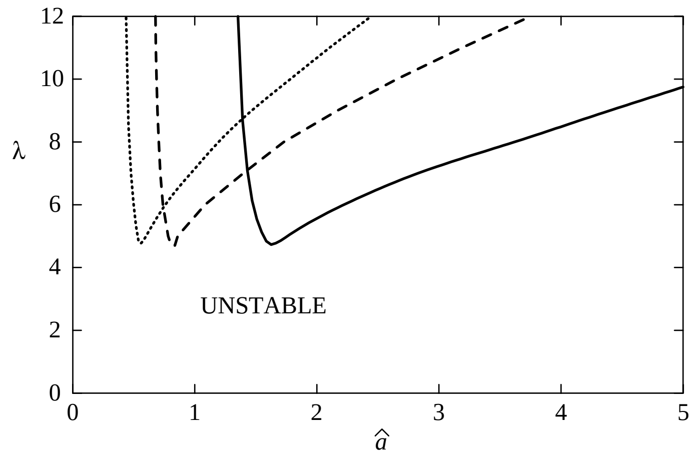

Figure 2. The neutral curve of the fundamental (solid line), second harmonic (dashed line) and third harmonic (dotted line) for the first Görtler mode.

solely concentrated on the so-called first mode since this mode has (by definition) the highest growth rates at most wavenumbers. In the follow-up studies by Dando (1992) and Blackaby & Choudhari (1993), it was shown (independently) that the higher modes, with increasing cross-flow, behave very similarly to the first mode but, for certain ranges of wavenumber, these higher modes can still be (marginally) unstable even though the first mode has been completely stabilized by the cross-flow. Since all these modes satisfy the same equation (2.8) (or the scaled version, equation (2.12)), the analysis presented in this article is relevant to all of these modes; however, for simplicity's sake, we restrict our attention to the first mode (i.e. the one considered by Bassom & Hall (1991)) when presenting numerical results of the evolution equation in § 6.

The close relationship between equation (2.8) and the Taylor–Goldstein equation was considered in detail by Blackaby & Choudhari (1993). In particular, they proposed a generalized definition of the Richardson number for such vortex flows, namely

$$J = \frac{a^2 K \overline{u}_c \overline{u}_{Y_c}}{(\beta \overline{u}_{Y_c} + \mathrm{i} a \lambda \overline{w}_{Y_c})^2}, \tag{2.13}$$

where the subscript 'c' denotes evaluation at the critical level, $Y = Y_c$, where

$$\beta \overline{u}(Y_c) + \mathrm{i} a \lambda \overline{w}(Y_c) = 0;$$

note that β is purely imaginary for these neutral modes. A similar definition of the Richardson number is available for inviscid Görtler modes in compressible three-dimensional boundary-layers flows (see Dando 1993).

It is possible to calculate neutral curves for the Görtler modes (see figure 2; see also figure 1 in Blackaby & Choudhari 1993) and then using the same definition of ν as used in the stratified shear-layer problem, namely

$$\nu^2 = \tfrac{1}{4}(1 - 4J), \tag{2.14}$$

we can calculate the numerical values of ν on the neutral curves. In fact, to fully determine ν, it is necessary to inspect the shape of the eigenfunctions of the neutral modes, so that the sign of ν can be deduced. It can easily be shown that solutions of (2.8) have the behaviour

$$\tilde{\tilde{V}} \sim |Y - Y_c|^{1/2+\nu} + O(|Y - Y_c|^{-1/2+\nu}) \quad \text{as} \quad Y \to Y_c,$$

i.e. near the critical-layer, the solution is proportional to just one of the Frobenius series solutions (i.e. the one corresponding to the indicial root $\frac{1}{2} + \nu$) and contains *none* of the other Frobenius series solution (corresponding to the indicial root $\frac{1}{2} - \nu$).

The last result has a quite significant effect on the analysis necessary to derive a nonlinear evolution-equation for the amplitude of the disturbance; essentially, a corollary of this result is that the 'largest' nonlinear jump term has zero coefficient and hence we have to seek non-zero, nonlinear jumps at higher orders in the critical-layer analysis. The property that neutral eigenfunctions of (2.8) are proportional to just one of the associated Frobenius solutions near the critical layer was noted by Blackaby & Choudhari (1993). However, given the similarities between (2.8) and the Taylor–Goldstein equation, this result is not surprising; it is merely an extension of one of the theorems due to Miles (1961).

In figure 2, the neutral curve for the first (fundamental) inviscid-Görtler-mode is presented. Recall that, in contrast to other hydrodynamic stability analyses, the inside of the neutral curve corresponds to stable flow configurations, i.e. on crossing the left-hand branch of the neutral curve (in the direction of increasing a) we move from a regime where the flow is unstable, into a regime where the flow is stable. The weakly nonlinear problem considered in this paper is appropriate to locations just to the left of the left-hand branch of this neutral curve. We consider disturbance(s) which are initially generated in the unstable regime (well to) the left of the left-hand branch of the neutral curve. The disturbance will initially grow (since the flow is unstable there) but viscous spreading effects (e.g. boundary-layer growth) will dampen the growth rate of the disturbance and the disturbance approaches a neutral state corresponding to a location on the left-hand branch of the neutral curve. However, before a neutral state can be attained (if it ever is), the disturbance must pass through the the weakly nonlinear regime located just to the left of the left-hand branch; note that weakly nonlinear theory is valid here since (i) viscous spreading effects have dampened the growth rate, and (ii) since the disturbance has evolved through an unstable machine, its amplitude is no longer infinitesimally small and so nonlinear effects cannot be ignored.

In fact, the effect of boundary-layer growth on the disturbance can be deduced immediately from the scaling (2.11a). Experimental studies show that the physical wavenumber a remains fixed as the Görtler vortices move downstream. In the downstream direction x increases and since $m - 1 < 0$, $\hat{a}$ must also increase. Thus, as mentioned above, the disturbance approaches a neutral state corresponding to a location on the left-hand branch of the neutral curve, but first it enters a weakly nonlinear regime located just to the left of the left-hand branch. However, since the disturbance is moving towards a neutral state (rather than away from one as assumed in the traditional Stuart–Watson–Landau type theory), the supercriticality is relatively large and so the weakly nonlinear analysis must consider a so-called non-equilibrium (i.e. x-dependent) critical-layer. It is found that $\nu < -\frac{1}{2}$ for the inviscid-Görtler-vortex neutral modes being studied here (see figure 3). Thus, based on the work of Blackaby *et al.* (1993) we can deduce that there are three possible nonlinear, integro-differential evolution equations for the non-equilibrium critical-layer regime: (i) one where the cubic-nonlinearity is directly due to viscous effects; (ii) another where the cubic-nonlinearity is due to supercriticality effects (i.e. because

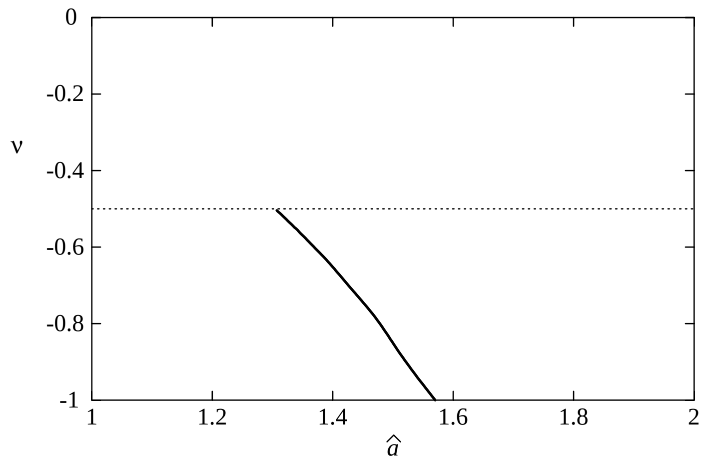

Figure 3. The values of ν for the left-hand branch of the neutral curve of the first Görtler mode.

the flow quantities are slightly away from their neutral values); and (iii), another where the nonlinearity is quintic.

Finally in this section, let us return to a discussion of figure 2; note that, as well as the neutral curve for the fundamental of the first inviscid Görtler mode (corresponding to the mode studied by Bassom & Hall (1991)), neutral curves have also been shown for the second and third harmonics of the *first* mode. To avoid confusion, it is important to note that the neutral curve for the second harmonic (say) is not the same as the neutral curve for the second mode as presented in figure 1 of Blackaby & Choudhari (1993). In the context of figure 2 and this discussion, the second and third harmonics are defined as solutions of (2.8), (2.12) with a replaced by $2a$ and $3a$ respectively (and similarly for β); nonlinear effects are neglected here as the disturbances are assumed to be sufficiently small. The physical motivation behind considering such higher harmonics is that, in any practical flow, the receptivity mechanisms responsible for the initiation of the disturbances will result in all harmonics of the first mode being generated (and hence present) in the flow. Thus, it is important to check that our problem is a sensible one to be studying (i.e. we must check that there are not more unstable/dangerous disturbances also present in the flow); the neutral curves presented in figure 2 reassure us that our problem is a sensible one to be considering. This is because the figure illustrates that, in the neighbourhood to the left of the left-hand branch of the neutral curve for the fundamental, the higher harmonics (even if they *still* exist to the right of their own neutral curves) are stable or, at worse, marginally unstable i.e. at the location under consideration in this study, the fundamental being considered has a comparable or larger growth rate than other disturbances which may be present in the flow. Similarly, figure 1 of Blackaby & Choudhari (1993) assures us that the fundamental of the first mode is at least as unstable as the higher modes at the location under consideration in this study. Moreover, it should be noted that, since the initial generation of disturbances will occur to the left of the neutral curves, the higher modes and harmonics would have evolved through their own nonlinear regimes (in the neighbourhood of the left-hand branches of their own neutral curves) and stable regimes; thus, there is no guarantee that these modes will still be present in the flow at the location under consideration in this study. This discussion is motivated by the recent conference paper by Timoshin (1996).

3. The nonlinear problem: the flow outside of the critical layer

In order to derive the desired evolution equations a study of the fundamental and other higher harmonics is necessary both inside and outside of the critical layer. The details of this study are dependent on the flow under consideration but the method is quite general and can be applied to other flows (the reader will note many similarities between the next two sections and the corresponding analysis in Blackaby *et al.* (1993) for the stratified shear layer problem where some aspects are discussed in more detail). In this section we consider the flow outside the critical layer while the following two sections are devoted to the flow inside the critical layer.

(*a*) *Scales and notation*

First, let us consider for a moment the various x-scales present in the current problem. In the analysis, it is necessary to consider three further x-scales, in addition to the boundary layer variable, x, such that the streamwise derivative has the form,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \mu^{-1}\frac{\partial}{\partial x_1} + G^{1/2}\mu\frac{\partial}{\partial \tilde{X}} + G^{1/2}\frac{\partial}{\partial X}, \tag{3.1}$$

where

(i) X is the fast scale implicit in the expansions (2.6) and is necessary for the derivation of the governing equation (2.8);

(ii) $\tilde{X}$ is a slower scale, over which the disturbance amplitude grows (μ is a small parameter which measures the size of the growth rate);

(iii) x_1 is the slowest of these three scales (provided $G^{-1/4} \ll \mu$), over which the mean flow varies.

Now, let us consider the notation used in this paper. The length of many of the equations in the critical-layer analysis necessitates that we use as much abbreviated notation as possible. In particular, the real functions $\overline{q}_0$, $\overline{q}_1$ and $\overline{q}_2$ are defined by

$$\mathrm{i}\overline{q}_0 = \beta\overline{u}_0 + \mathrm{i}a_0\lambda_0\overline{w}_0, \tag{3.2 a}$$

$$\mathrm{i}\overline{q}_1 = \beta\overline{u}_1 + \mathrm{i}a_0(\lambda_1\overline{w}_0 + \lambda_0\overline{w}_1) + \mathrm{i}a_1\lambda_0\overline{w}_0, \tag{3.2 b}$$

$$\mathrm{i}\overline{q}_2 = \beta\overline{u}_2 + \mathrm{i}a_0(\lambda_2\overline{w}_0 + \lambda_1\overline{w}_1 + \lambda_0\overline{w}_2) + \mathrm{i}a_1(\lambda_1\overline{w}_0 + \lambda_0\overline{w}_1) + \mathrm{i}a_2\lambda_0\overline{w}_0, \tag{3.2 c}$$

where, for instance, $\overline{u}_0 = \overline{u}(x_0)$, with $\overline{u}$ given by equation (2.9) and x_0 being the x-location under consideration. Note that the critical layer is located, at $Y = Y_c$, where $\overline{q}_0 = 0$ (i.e. where equation (2.8) is singular). Also $\overline{u}_1$, $\overline{u}_2$, $\overline{w}_1$, $\overline{w}_2$, λ_1 and λ_2 are real functions of x_1 and arise from Taylor-series expansions of the quantities, for small x_1, about the location $x = x_0$; for example

$$\overline{u}_1 = x_1\overline{u}_{0x}(x_0) \quad \text{and} \quad \lambda_2 = x_1^2\lambda_{0xx}(x_0)/2. \tag{3.3}$$

Similarly the perturbation to the curvature, K_1, and to the spanwise wavenumber, a_1, are real functions of x_1. Henceforth, dashes on mean flow quantities shall denote derivatives with respect to Y. It is also implicit that all of the mean flow quantities which occur in the critical layer analysis (§ 3 *c*, § 4, § 5 and § 6) are their values evaluated at the critical layer. We shall not explicitly write a subscript c to denote this.

(*b*) *Formulation and the solvability condition*

As indicated in the previous section, inviscid Görtler vortices arise in boundary-layer flows, over concave surfaces, at large values of the Reynolds number (Re) and

Görtler number (G). In the rest of this paper, it is assumed that $Re \gg G \gg 1$. The linear problem has been extensively studied for several different flows; in this study we (begin to) address the so-called nonlinear problem, i.e. how do the growing inviscid Görtler vortices (based on linear theory) behave/evolve when nonlinear effects are physically important (and hence included in the mathematical analysis)?

The aim of the nonlinear problem/analysis is to derive an evolution equation which governs the subsequent amplitude of the vortex disturbance (recall that the linear problem does not fix this amplitude, instead it is merely assumed that the disturbance is infinitesimally small). As typical in such nonlinear-critical-layer studies, the evolution equation arises by matching a so-called nonlinear jump (stemming from the analysis of the critical-layer region itself) with a so-called solvability condition which stems from a study of the flow away from the critical-layer. Thus, the aim of this section is to derive the solvability condition; to do so requires studying additional terms in the asymptotic expansions of the velocity and pressure disturbances (rather than just the largest fundamental) and it is therefore sensible to spend a few moments considering the formulation.

The total flow (i.e. the three-dimensional boundary-layer flow plus the inviscid-vortex disturbance) is written

$$\left.\begin{aligned} \underline{u} &= u_\infty^*(\overline{u}, Re^{-1/2}\overline{v}, Re^{-1/2}G^{1/2}\lambda\overline{w}) \\ &\qquad + u_\infty^*(U, Re^{-1/2}G^{1/2}V, Re^{-1/2}G^{1/2}W), \\ p &= \overline{p} + Re^{-1}GP. \end{aligned}\right\} \tag{3.4}$$

Then, the vortex components of the total flow are expanded as a sum of their harmonics, e.g. the normal component of the disturbance is written

$$V(x, Y, Z) = \sum_{l=-\infty}^{\infty} V_l(x_0, \tilde{X}, Y)\exp\{l(\beta X + \mathrm{i}aZ)\}, \tag{3.5}$$

with similar expansions for U, W and P; note that l denotes the harmonic. In the analysis, is necessary to consider, in particular, the fundamental ($l = \pm 1$), zeroth ($l = 0$) and second harmonic ($l = \pm 2$) terms. In fact, the fundamental is expanded as

$$V_1 = \epsilon V_1^{(1)} + \epsilon\mu V_1^{(2)} + \ldots; \tag{3.6 a}$$

the zeroth and second harmonics are expanded as

$$V_0 = \epsilon^2 V_0^{(1)} + \ldots, \qquad V_2 = \epsilon^2 V_2^{(1)} + \ldots, \tag{3.6 b}$$

together with similar expansions for U_l, W_l and P_l.

Let us consider the terms in the expansion of the fundamental V_1; we note that $V_1^{(1)}$ corresponds to the neutral mode of the inviscid linear problem, while $V_1^{(2)}$ is the correction to account for the $\tilde{X}$-dependence of the solution. Thus $V_1^{(1)}$ satisfies

$$L_1 V_1^{(1)} = 0; \qquad V_1^{(1)}(0) = 0, \quad V_1^{(1)} \to 0 \quad \text{as} \quad Y \to \infty, \tag{3.7}$$

where

$$L_l \equiv \frac{\partial^2}{\partial Y^2} - l^2 a_0^2 - \frac{\overline{q}_0''}{\overline{q}_0} + J_0\left(\frac{\overline{q}_0'}{\overline{q}_0}\right)^2 \quad \text{with} \quad J_0(x_0, Y) = -\frac{a_0^2 K_0 \overline{u}_0 \overline{u}_0'}{\overline{q}_0'^2}. \tag{3.8}$$

In fact, for definiteness, we write

$$V_1^{(1)} = B_\pm(\tilde{X}) V_a(Y), \tag{3.9 a}$$

where the amplitude $B(\tilde{X})$ is the function whose properties/evolution are the goal of our analysis, while the normalized eigenfunction $V_a(Y)$ satisfies (3.8) together with the property that

$$V_a(Y) = |Y - Y_c|^{1/2+\nu} + \dots \quad \text{as} \quad (Y - Y_c) \to 0. \tag{3.9 b}$$

The + and − signs on B denote, respectively, above and below the critical layer.

The second term in the expansion of V_1 satisfies the equation

$$L_1 V_1^{(2)} = Q_1, \tag{3.10}$$

subject to the boundary conditions $V_1^{(2)}(Y = 0) = 0$ and $V_1^{(2)} \to 0$ as $Y \to \infty$. The function Q_1 can be written

$$Q_1 = Q_{11} V_1^{(1)} + Q_{12} V_{1\tilde{X}}^{(1)}, \tag{3.11 a}$$

where

$$Q_{11} = 2a_0 a_1 + \frac{\overline{q}_1''}{\overline{q}_0} - \frac{\overline{q}_1 \overline{q}_0''}{\overline{q}_0^2} - J_0 \left(2\frac{a_1}{a_0} + \frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0} - 2\frac{\overline{q}_1}{\overline{q}_0} \right) \frac{\overline{q}_0'^2}{\overline{q}_0^2}, \tag{3.11 b}$$

$$Q_{12} = \frac{\mathrm{i}\overline{u}_0}{\overline{q}_0} \left(\frac{\overline{q}_0''}{\overline{q}_0} - \frac{\overline{u}_0''}{\overline{u}_0} - 2J_0 \frac{\overline{q}_0'^2}{\overline{q}_0^2} \right). \tag{3.11 c}$$

The solution to this equation can be considered to be the sum,

$$V_1^{(2)} = V_{1PI}^{(2)} + V_{1CF}^{(2)},$$

of a particular integral, $V_{1PI}^{(2)}$, and the complementary function, $V_{1CF}^{(2)}$. As $Y - Y_c \to 0$,

$$\begin{aligned} V_{1CF}^{(2)} = {} & B_\pm a_{1\pm}^{(2)} |Y - Y_c|^{1/2+\nu} (1 + O(|Y - Y_c|^{-1})) \\ & + B_\pm b_{1\pm}^{(2)} |Y - Y_c|^{1/2-\nu} (1 + O(|Y - Y_c|^{-1})), \end{aligned} \tag{3.12}$$

where $a_{1\pm}^{(2)}$ and $b_{1\pm}^{(2)}$ are constants as yet undetermined. Note that if the Frobenius roots, $\frac{1}{2} \pm |\nu|$ differ by an integer then equation (3.12) is no longer appropriate (logarithms are needed). As such cases ($\nu = \frac{1}{2}m$; m an integer) are isolated, we choose not to concern ourselves with them (and their immediate neighbourhood) in this paper.

A solvability condition for the above boundary-value problem is required. Note that: (i) the operator L_1 is self-adjoint away from the critical level $Y = Y_c$ (where $\overline{q}_0 = 0$), and (ii) the right-hand side of (3.10) is singular at $Y = Y_c$. Following the method of Hickernell (1984), the solvability condition is derived by multiplying both sides of equation (3.10) by $V_1^{(1)}$ and integrating over all Y, excluding the (sole) critical layer at $Y = Y_c$. After integrating by parts; imposing the boundary conditions at $Y = 0, \infty$; and the asymptotic forms of $V_1^{(1)}$ and $V_{1CF}^{(2)}$ as $Y \to Y_c$, it follows that

$$⨍_0^\infty V_1^{(1)} Q_1 \, \mathrm{d}Y = -[V_1^{(1)} V_{1CF}^{(2)'} - V_1^{(1)'} V_{1CF}^{(2)}]_{Y_c-}^{Y_c+} = 2\nu B^2 (b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)}), \tag{3.13}$$

where the barred integral signs denote the finite parts of these integrals and we have used the relationships $B_- = \mathrm{i}^{-1-2\nu} B_+$, $B_+ \equiv B$ (see § 4 *a*). After substituting for Q_1, the solvability condition can be written

$$(I_3 - \mathrm{i}^{-4\nu} I_1) \frac{\partial B}{\partial \tilde{X}} + (I_4 - \mathrm{i}^{-4\nu} I_2) B = 2\nu B (b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)}), \tag{3.14}$$

where I_1, I_2, I_3 and I_4 are defined as follows:

$$I_1 = \int_0^{Y_c} Q_{12} V_a^2 \,\mathrm{d}Y, \tag{3.15 a}$$

$$I_2 = \int_0^{Y_c} Q_{11} V_a^2 \,\mathrm{d}Y, \tag{3.15 b}$$

$$I_3 = \int_{Y_c}^{\infty} Q_{12} V_a^2 \,\mathrm{d}Y, \tag{3.15 c}$$

$$I_4 = \int_{Y_c}^{\infty} Q_{11} V_a^2 \,\mathrm{d}Y. \tag{3.15 d}$$

These integrals need to be evaluated numerically. In order to identify the singular parts close to the critical layer, we substitute in the known asymptotic forms of the integrands and integrate by parts.

(c) *The asymptotic expansions as* $(Y - Y_c) \to 0$

In terms of the critical layer variable $\eta = \mu^{-1}(Y - Y_c)$, the asymptotes for V and Φ, as $(Y - Y_c) \to 0$, where the function Φ is given by

$$\Phi_l = V_l - \frac{2la_0^2}{(1-2\nu)\mathrm{i}\overline{q}_0'} P_l, \tag{3.16}$$

(l denotes the harmonic), for the fundamental, zeroth and second harmonics are

$$\begin{aligned} V_1 = {} & \epsilon\mu^{1/2+\nu}(B_\pm|\eta|^{1/2+\nu} + \dots) \\ & + \epsilon\mu^{3/2+\nu}\left(B_\pm|\eta|^{3/2+\nu}\left[\frac{\overline{q}_0''}{\overline{q}_0'(1+2\nu)} + \frac{(1-2\nu)}{4}\left(\frac{\overline{q}_0''}{\overline{q}_0'} - \frac{\overline{u}_0''}{\overline{u}_0'} - \frac{\overline{u}_0'}{\overline{u}_0}\right)\right]\right. \\ & \left. + B_\pm a_{1\pm}^{(2)}|\eta|^{1/2+\nu} + \dots\right) + \epsilon\mu^{3/2-\nu}B_\pm|\eta|^{1/2-\nu}b_{1\pm}^{(2)} + \dots, \end{aligned} \tag{3.17 a}$$

$$\begin{aligned} \Phi_1 = {} & \epsilon\mu^{1/2+\nu}.0 + \epsilon\mu^{3/2+\nu}\left(B_\pm|\eta|^{3/2+\nu}\left[\frac{\overline{q}_0''}{(1+2\nu)\overline{q}_0'} - \frac{\overline{u}_0''}{2\overline{u}_0'} - \frac{\overline{u}_0'}{2\overline{u}_0}\right]\right) \\ & - \epsilon\mu^{3/2-\nu}|\eta|^{1/2-\nu}b_{1\pm}^{(2)}B_\pm\frac{4\nu}{(1-2\nu)} + \dots, \end{aligned} \tag{3.17 b}$$

$$V_0 = \epsilon^2\mu^{-1+2\nu}.0 + \dots, \tag{3.18}$$

$$V_2 = \epsilon^2\mu^{-1+2\nu}\left(\frac{\mathrm{i}(1+2\nu)}{\overline{q}_0'(3-2\nu)}B_\pm^2|\eta|^{-1+2\nu} + \dots\right) + \dots, \tag{3.19 a}$$

$$\Phi_2 = \epsilon^2\mu^{-1+2\nu}.0 + \dots. \tag{3.19 b}$$

Note that the coefficient of the leading terms in the Φ_1, V_0 and Φ_2 expansions are all zero; this is a consequence of the property that the neutral eigenfunctions of (2.8) are proportional to just one of the associated Frobenius solutions near the critical layer (Blackaby & Choudhari 1993; Miles 1961).

4. The critical layer

The main purpose of this section is to derive a second relation between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ (the first being given by the solvability condition derived in the last section, equation (3.14)) in order to obtain the desired nonlinear evolution equation.

On writing

$$\left.\begin{aligned} \underline{u} &= u_\infty^*(\overline{u}, Re^{-1/2}\overline{v}, Re^{-1/2}G^{1/2}\lambda\overline{w}) \\ &\qquad + u_\infty^*(\mu^{-1}\hat{U}, Re^{-1/2}G^{1/2}\hat{V}, Re^{-1/2}G^{1/2}\mu^{-1}\hat{W}), \\ p &= \overline{p} + Re^{-1}G\hat{P}, \end{aligned}\right\} \tag{4.1}$$

where $\hat{U}$, $\hat{V}$, $\hat{W}$ and $\hat{P}$ are functions of X, $\tilde{X}$, Z and the critical-layer normal variable

$$\eta = \mu^{-1}(Y - Y_{\rm c}), \tag{4.2}$$

the governing equations for the vortex disturbance in the critical layer can be written

$$\beta l\hat{U} + \hat{V}_\eta + {\rm i}a_0 l\hat{W} = -\mu(\hat{U}_{\tilde{X}} + {\rm i}a_1 l\hat{W}), \tag{4.3 a}$$

$$\begin{aligned} \overline{u}_0\hat{U}_{\tilde{X}} + {\rm i}(\overline{q}_0'\eta + \overline{q}_1)l\hat{U} + \overline{u}_0'\hat{V} &= -\mu^{-2}(\hat{U}\hat{U}_X + \mu\hat{U}\hat{U}_{\tilde{X}} + \hat{V}\hat{U}_\eta + \hat{W}\hat{U}_Z) \\ &\quad -\mu(\{\overline{u}_0'\eta + \overline{u}_1\}\hat{U}_{\tilde{X}} + {\rm i}\{\tfrac{1}{2}\overline{q}_0''\eta^2 + \overline{q}_1'\eta + \overline{q}_2\}l\hat{U} \\ &\quad + \{\overline{u}_0''\eta + \overline{u}_1'\}\hat{V}) + G^{-1/2}\mu^{-3}\hat{U}_{\eta\eta}, \end{aligned} \tag{4.3 b}$$

$$K_0\overline{u}_0\hat{U} + \hat{P}_\eta = -\mu^{-1}K_0\hat{U}^2/2 - \mu(K_0\overline{u}_0'\eta + K_1\overline{u}_0 + K_0\overline{u}_1)\hat{U}, \tag{4.3 c}$$

$$\begin{aligned} \overline{u}_0\hat{W}_{\tilde{X}} + {\rm i}(\overline{q}_0'\eta + \overline{q}_1)l\hat{W} + \lambda_0\overline{w}_0'\hat{V} + {\rm i}a_0 l\hat{P} &= -\mu^{-2}(\hat{U}\hat{W}_X + \mu\hat{U}\hat{W}_{\tilde{X}} + \hat{V}\hat{W}_\eta + \hat{W}\hat{W}_Z) \\ &\quad -\mu(\{\overline{u}_0'\eta + \overline{u}_1\}\hat{W}_{\tilde{X}} + {\rm i}\{\tfrac{1}{2}\overline{q}_0''\eta^2 + \overline{q}_1'\eta + \overline{q}_2\}l\hat{W} \\ &\quad + \{\lambda_0\overline{w}_0''\eta + \lambda_1\overline{w}_0' + \lambda_0\overline{w}_1'\}\hat{V} + {\rm i}a_1 l\hat{P}) + G^{-1/2}\mu^{-3}\hat{W}_{\eta\eta}, \end{aligned} \tag{4.3 d}$$

where we have retained on the right-hand sides the leading order effects due to non-linearity, viscosity and terms leading to a perturbation of the generalized Richardson number of Blackaby & Choudhari (1993), see equation (2.13). Recall that dashes on mean flow quantities denote derivatives with respect to Y; moreover, it is implicit that all of the mean flow quantities which occur here and later in the critical layer analysis actually correspond to their values evaluated at the critical layer, i.e. to simplify the notation we omit the subscript 'c' on mean flow quantities.

In this section we shall solve the governing equations for the relevant higher order terms of the harmonics. As suggested by the work of Blackaby *et al.* (1993) we consider in this initial work two evolution equations; one where the cubic-nonlinearity is due to supercriticality (non-neutrality) effects, and another where the cubic-nonlinearity it is directly due to viscous effects. In this section we shall concentrate on deriving the first of these (the equivalent of the so-called J_1-cubic of the previous paper). In order to do this it is helpful to introduce the operator

$$\hat{N}_{l,\chi} = \left(\overline{u}_0\frac{\partial}{\partial\tilde{X}} + {\rm i}l(\overline{q}_0'\eta + \overline{q}_1)\right)\frac{\partial}{\partial\eta} - {\rm i}l\chi\overline{q}_0', \tag{4.4}$$

where l again denotes the harmonic. We have assumed that $G^{-1/2}\mu^{-3} \ll 1$, i.e. that viscous effects are not large enough to affect the operator, $\hat{N}_{l,\chi}$, at leading order.

The solution for the vortex-flow inside the critical layer is also constructed in the form of a Fourier series, i.e. the normal component of the velocity is written:

$$\hat{V}(x_0, X, \tilde{X}, \eta, Z) = \sum_{l=-\infty}^{\infty} \hat{V}_l(x_0, \tilde{X}, \eta) \exp\{l(\beta X + \mathrm{i}aZ)\}, \tag{4.5}$$

and the fundamental, zeroth and second harmonics, respectively, are then expanded as asymptotic series:

$$\hat{V}_1 = \epsilon\mu^{1/2+\nu}\hat{V}_1^{(1)} + \ldots + \epsilon^3\mu^{-5/2+3\nu}\hat{V}_1^{(2)} + \ldots + \epsilon\mu^{3/2+\nu}\hat{V}_1^{(3a)} + \ldots$$
$$+\epsilon\mu^{3/2-\nu}\hat{V}_1^{(3b)} + \ldots + \epsilon^3\mu^{-3/2+3\nu}\hat{V}_1^{(4)} + \ldots, \tag{4.6 a}$$
$$\hat{V}_0 = \epsilon^2\mu^{-1+2\nu}\hat{V}_0^{(1)} + \ldots + \epsilon^2\mu^{2\nu}\hat{V}_0^{(2)} + \ldots, \tag{4.6 b}$$
$$\hat{V}_2 = \epsilon^2\mu^{-1+2\nu}\hat{V}_2^{(1)} + \ldots + \epsilon^2\mu^{2\nu}\hat{V}_2^{(2)} + \ldots, \tag{4.6 c}$$

and similarly for $\hat{U}$, $\hat{W}$, $\hat{P}$ and the important functions

$$\hat{\Phi}_l = \hat{V}_l - \frac{2la_0^2}{(1-2\nu)\mathrm{i}\overline{q}_0'}\hat{P}_l. \tag{4.7}$$

These expansions are not necessarily completely ordered (depending on the relative sizes of ϵ and μ) and moreover we have only retained the terms necessary for deriving the desired evolution equation. The scalings follow directly from the outer asymptotes and/or by considering the process of harmonic generation.

(a) $O(\epsilon\mu^{1/2+\nu})$ of the fundamental

At this order the governing equations are

$$\beta\hat{U}_1^{(1)} + \hat{V}_{1\eta}^{(1)} + \mathrm{i}a_0\hat{W}_1^{(1)} = 0, \tag{4.8 a}$$
$$\overline{u}_0\hat{U}_{1\tilde{X}}^{(1)} + \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{U}_1^{(1)} + \overline{u}_0'\hat{V}_1^{(1)} = 0, \tag{4.8 b}$$
$$K_0\overline{u}_0\hat{U}_1^{(1)} + \hat{P}_{1\eta}^{(1)} = 0, \tag{4.8 c}$$
$$\overline{u}_0\hat{W}_{1\tilde{X}}^{(1)} + \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_1^{(1)} + \lambda_0\overline{w}_0'\hat{V}_1^{(1)} + \mathrm{i}a_0\hat{P}_1^{(1)} = 0; \tag{4.8 d}$$

these can be manipulated to give

$$\hat{N}_{1,1/2+\nu}\hat{V}_1^{(1)} = \tfrac{1}{2}(1-2\nu)\mathrm{i}\overline{q}_0'\hat{\Phi}_1^{(1)}, \qquad \hat{N}_{1,1/2-\nu}\hat{\Phi}_1^{(1)} = 0, \tag{4.9}$$

with solutions (which match to the corresponding outer solutions)

$$\hat{V}_1^{(1)}(\tilde{X}, \eta) = \frac{(1+2\nu)\mathrm{i}^{3/2-\nu}}{2\Gamma(\frac{1}{2}-\nu)}\left(\frac{\overline{u}_0}{\overline{q}_0'}\right)^{1/2+\nu}$$
$$\times \int_0^\infty \mathrm{d}x_1 B(\tilde{X} - x_1)x_1^{-3/2-\nu}\exp\{-\mathrm{i}x_1(\overline{q}_0'\eta + \overline{q}_1)/\overline{u}_0\}, \tag{4.10 a}$$
$$\hat{\Phi}_1^{(1)}(\tilde{X}, \eta) = 0. \tag{4.10 b}$$

A detailed discussion of the solution of equations of the type (4.9a, b) can be found in the studies of Churilov & Shukhman (1988) and Dando (1993). Note, however, that since $\nu < -\frac{1}{2}$ for the inviscid Görtler problem, there is no need to evaluate the integral occurring in expression (4.10) around a complex contour (cf. the stratified shear-flow problem). The function $\hat{V}_1^{(1)}(\tilde{X},\eta)$ has a single asymptotic representation in the lower-half plane $(-\pi \leqslant \arg\eta \leqslant 0)$

$$\hat{V}_1^{(1)}(\tilde{X},\eta) = B(\tilde{X})\eta^{1/2+\nu} + O(\eta^{-1/2+\nu}) \quad \text{as} \quad |\eta| \to \infty, \tag{4.11}$$

and matching with the outer asymptote (3.17*a*) fixes

$$B_+(\tilde{X}) \equiv B(\tilde{X}) \quad \text{and} \quad B_-(\tilde{X}) = \mathrm{i}^{-1-2\nu}B(\tilde{X}). \tag{4.12}$$

Later we shall derive evolution equations for the amplitude, $B(\tilde{X})$, but for the moment we can regard it as an arbitrary function that satisfies the requirement $B(\tilde{X}) \to 0$ as $\tilde{X} \to -\infty$. This requirement is consistent with the initial condition used for our evolution equation (see § 6), which itself is a result of insisting that the solution of the evolution equation matches to an 'earlier' linear stage.

(*b*) $O(\epsilon^2\mu^{-1+2\nu})$ *of the second harmonic*

At this order, equations (4.3 *a*)–(4.3 *d*) yield

$$2\beta\hat{U}_2^{(1)} + \hat{V}_{2\eta}^{(1)} + 2\mathrm{i}a_0\hat{W}_2^{(1)} = 0, \tag{4.13 a}$$

$$\overline{u}_0\hat{U}_{2\tilde{X}}^{(1)} + 2\mathrm{i}(\overline{q}_0'\eta+\overline{q}_1)\hat{U}_2^{(1)} + \overline{u}_0'\hat{V}_2^{(1)} = -(\beta\hat{U}_1^{(1)}\hat{U}_1^{(1)} + \hat{V}_1^{(1)}\hat{U}_{1\eta}^{(1)} + \mathrm{i}a_0\hat{W}_1^{(1)}\hat{U}_1^{(1)}), \tag{4.13 b}$$

$$K_0\overline{u}_0\hat{U}_2^{(1)} + \hat{P}_{2\eta}^{(1)} = 0, \tag{4.13 c}$$

$$\overline{u}_0\hat{W}_{2\tilde{X}}^{(1)} + 2\mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_2^{(1)} + \lambda_0\overline{w}_0'\hat{V}_2^{(1)} + \mathrm{i}a_0\hat{P}_2^{(1)} = -(\beta\hat{U}_1^{(1)}\hat{W}_1^{(1)} + \hat{V}_1^{(1)}\hat{W}_{1\eta}^{(1)} + \mathrm{i}a_0\hat{W}_1^{(1)}\hat{W}_1^{(1)}), \tag{4.13 d}$$

from which it can be shown that $\hat{V}_2^{(1)}$ and $\hat{\Phi}_2^{(1)}$ satisfy

$$\hat{N}_{2,1/2+\nu}\hat{V}_2^{(1)} = (1-2\nu)\mathrm{i}\overline{q}_0'\hat{\Phi}_2^{(1)} + 2(\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta}^{(1)} - \hat{V}_1^{(1)}\hat{V}_{1\eta\eta}^{(1)}), \quad \hat{N}_{2,1/2-\nu}\hat{\Phi}_2^{(1)} = 0. \tag{4.14 a, b}$$

Since the right-hand sides of (4.14 *a*, *b*) do not involve the conjugate of $\hat{V}_1^{(1)}$, it follows that $\hat{V}_2^{(1)}$ and $\hat{\Phi}_2^{(1)}$ have unique asymptotic representations as $|\eta| \to \infty$ (in the lower half plane of complex η). The solutions of these equations, which match to the corresponding solutions outside of the critical layer, are

$$\begin{aligned}\hat{V}_2^{(1)} = \frac{\mathrm{i}^{-2\nu}(1+2\nu)^2}{4\overline{u}_0\Gamma^2(\frac{1}{2}-\nu)}\left(\frac{\overline{u}_0}{\overline{q}_0'}\right)^{2\nu}\int_0^\infty \mathrm{d}\tilde{x}_3\int_0^\infty \mathrm{d}\tilde{x}_1\int_0^\infty \mathrm{d}\tilde{x}_2\, B(\tilde{X}-\tilde{x}_3-\tilde{x}_1)B(\tilde{X}-\tilde{x}_3-\tilde{x}_2)\\ \times(\tilde{x}_1\tilde{x}_2)^{-3/2-\nu}(\tilde{x}_1-\tilde{x}_2)^2(\tilde{x}_1+\tilde{x}_2)^{1/2+\nu}(2\tilde{x}_3+\tilde{x}_1+\tilde{x}_2)^{-3/2-\nu}\\ \times\exp\{-\mathrm{i}(2\tilde{x}_3+\tilde{x}_1+\tilde{x}_2)(\overline{q}_0'\eta+\overline{q}_1)/\overline{u}_0\},\end{aligned} \tag{4.15 a}$$

$$\hat{\Phi}_2^{(1)} = 0; \tag{4.15 b}$$

(again, see Churilov & Shukhman (1988) and Dando (1993) for a discussion of the solution of equations like (4.14*a*) which have a non-zero right-hand side).

(*c*) $O(\epsilon^2\mu^{-1+2\nu})$ *of the zeroth harmonic*

At this order, the governing equations for the zeroth harmonic are

$$\hat{V}_{0\eta}^{(1)} = 0, \tag{4.16 a}$$

$$\overline{u}_0\hat{U}_{0\tilde{X}}^{(1)} + \overline{u}_0'\hat{V}_0^{(1)} = -(\hat{V}_1^{(1)}\hat{U}_{-1\eta}^{(1)} + \hat{V}_{-1}^{(1)}\hat{U}_{1\eta}^{(1)} - \mathrm{i}a_0\hat{W}_1^{(1)}\hat{U}_{-1}^{(1)} + \mathrm{i}a_0\hat{W}_{-1}^{(1)}\hat{U}_1^{(1)}), \tag{4.16 b}$$

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$$K_0\overline{u}_0\hat{U}_0^{(1)} + \hat{P}_{0\eta}^{(1)} = 0, \tag{4.16 c}$$

$$\overline{u}_0\hat{W}_{0\tilde{X}}^{(1)} + \lambda_0\overline{w}_0'\hat{V}_0^{(1)} = -(-\beta\hat{U}_1^{(1)}\hat{W}_{-1}^{(1)} + \beta\hat{U}_{-1}^{(1)}\hat{W}_1^{(1)} + \hat{V}_1^{(1)}\hat{W}_{-1\eta}^{(1)} + \hat{V}_{-1}^{(1)}\hat{W}_{1\eta}^{(1)}), \tag{4.16 d}$$

where the notation $\hat{V}_{-1}^{(1)}$, for example, denotes the complex conjugate of $\hat{V}_1^{(1)}$. These can be solved to give

$$\hat{U}_0^{(1)} = \frac{4\overline{u}_0'}{\overline{q}_0'^2(1+2\nu)^2}(\hat{V}_{1\eta}^{(1)}\hat{V}_{-1\eta}^{(1)})_\eta, \tag{4.17 a}$$

$$\hat{V}_0^{(1)} = 0, \tag{4.17 b}$$

$$\hat{W}_0^{(1)} = \frac{\mathrm{i}}{a_0}\left(\beta - \frac{(1+2\nu)\mathrm{i}\overline{q}_0'}{2\overline{u}_0'}\right)\hat{U}_0^{(1)}. \tag{4.17 c}$$

(d) $O(\epsilon^3\mu^{-5/2+3\nu})$ *of the fundamental*

At this order the largest nonlinear term emerges; in fact, in many critical-layer problems the desired nonlinear-jump would be found at this stage. However, here, the four governing equations are

$$\beta\hat{U}_1^{(2)} + \hat{V}_{1\eta}^{(2)} + \mathrm{i}a_0\hat{W}_1^{(2)} = 0, \tag{4.18 a}$$

$$\begin{aligned}\overline{u}_0\hat{U}_{1\tilde{X}}^{(2)} &+ \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{U}_1^{(2)} + \overline{u}_0'\hat{V}_1^{(2)} \\ &= -(\beta\hat{U}_2^{(1)}\hat{U}_{-1}^{(1)} + \beta\hat{U}_0^{(1)}\hat{U}_1^{(1)} + \hat{V}_2^{(1)}\hat{U}_{-1\eta}^{(1)} + \hat{V}_{-1}^{(1)}\hat{U}_{2\eta}^{(1)} + \hat{V}_1^{(1)}\hat{U}_{0\eta}^{(1)} \\ &\quad + \hat{V}_0^{(1)}\hat{U}_{1\eta}^{(1)} - \mathrm{i}a_0\hat{W}_2^{(1)}\hat{U}_{-1}^{(1)} + 2\mathrm{i}a_0\hat{W}_{-1}^{(1)}\hat{U}_2^{(1)} + \mathrm{i}a_0\hat{W}_0^{(1)}\hat{U}_1^{(1)}),\end{aligned} \tag{4.18 b}$$

$$K_0\overline{u}_0\hat{U}_1^{(2)} + \hat{P}_{1\eta}^{(2)} = 0, \tag{4.18 c}$$

$$\begin{aligned}\overline{u}_0\hat{W}_{1\tilde{X}}^{(2)} &+ \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_1^{(2)} + \lambda_0\overline{w}_0'\hat{V}_1^{(2)} + \mathrm{i}a\hat{P}_1^{(2)} \\ &= -(-\beta\hat{U}_2^{(1)}\hat{W}_{-1}^{(1)} + 2\beta\hat{U}_{-1}^{(1)}\hat{W}_2^{(1)} + \beta\hat{U}_0^{(1)}\hat{W}_1^{(1)} + \hat{V}_2^{(1)}\hat{W}_{-1\eta}^{(1)} + \hat{V}_{-1}^{(1)}\hat{W}_{2\eta}^{(1)} \\ &\quad + \hat{V}_1^{(1)}\hat{W}_{0\eta}^{(1)} + \hat{V}_0^{(1)}\hat{W}_{1\eta}^{(1)} + \mathrm{i}a_0\hat{W}_2^{(1)}\hat{W}_{-1}^{(1)} + \mathrm{i}a_0\hat{W}_0^{(1)}\hat{W}_1^{(1)}),\end{aligned} \tag{4.18 d}$$

from which it follows that $\hat{V}_1^{(2)}$ and $\hat{\Phi}_1^{(2)}$ satisfy

$$\begin{aligned}\hat{N}_{1,1/2+\nu}\hat{V}_1^{(2)} &= \tfrac{1}{2}(1-2\nu)\mathrm{i}\overline{q}_0'\hat{\Phi}_1^{(2)} - \tfrac{1}{2}(\hat{V}_{-1}^{(1)}\hat{V}_{2\eta}^{(1)})_\eta + \hat{V}_2^{(1)}\hat{V}_{-1\eta\eta}^{(1)} - \hat{V}_0^{(1)}\hat{V}_{1\eta\eta}^{(1)} \\ &\quad + \frac{2\mathrm{i}}{\overline{q}_0'(1+2\nu)}(\hat{V}_1^{(1)}(\hat{V}_1^{(1)}\hat{V}_{1\eta}^{(1)})_{\eta\eta} - \hat{V}_{1\eta}^{(1)}(\hat{V}_1^{(1)}\hat{V}_{1\eta}^{(1)})_\eta),\end{aligned} \tag{4.19 a}$$

$$\hat{N}_{1,1/2-\nu}\hat{\Phi}_1^{(2)} = 0. \tag{4.19 b}$$

The solution of (4.19 *b*), which matches to the corresponding solution outside of the critical layer, is

$$\hat{\Phi}_1^{(2)} = 0. \tag{4.20}$$

The last result can be regarded as a consequence of the property that the neutral eigenfunctions of (2.8) are proportional to just one of the associated Frobenius solutions near the critical layer. Thus, no nonlinear jump in $\hat{\Phi}_1$ arises at this order and we must seek a nonlinear jump at higher orders (note that a non-zero cubic

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nonlinear-jump typically does appear at this stage in weakly nonlinear analysis). However, here the lack of a nonlinear jump at this order is not surprising as the same outcome is found in the associated study of disturbances in a stratified shear flow (see Churilov & Shukhman (1988) and references therein). The fact that we have to seek a nonlinear jump in $\hat{\Phi}_1$ at higher orders in the critical-layer analysis leads to the nonlinear critical-layer analysis being more involved/complicated than one might otherwise expect. Moreover, it is found that there are three possible candidates competing for the role of largest nonlinear jump in $\hat{\Phi}_1$, corresponding to three different evolution equations for the disturbance amplitude $B(\tilde{X})$, depending on the relative sizes of the disturbance size ϵ, the criticality of the disturbance μ, and the viscous-effects as characterized by the Görtler number G.

(*e*) $O(\epsilon\mu^{3/2+\nu})$ *of the fundamental*

At this order, terms on the right-hand sides of the governing equations due to the perturbation of the 'Richardson number' first arise. However, the situation is much more complicated than for the stratified shear flow case as instead of just perturbing the Richardson number we now have to perturb the quantities in our generalized Richardson number, equation (2.13). We also have $\tilde{X}$-derivatives of previous critical-layer terms and higher order corrections to the base flow values appearing on the right-hand sides. Neither of these two effects were present for the stratified shear layer case considered in Blackaby *et al.* (1993) and they are a result of considering a spatial as opposed to a temporal evolution problem. These effects combined with the need to perturb $\overline{u}$, $\overline{w}$, a, λ and K, result in most equations and solutions in the present critical-layer analysis being considerably more lengthy than their counterparts for the stratified shear flow problem. Specifically, equations (4.3) give

$$\beta\hat{U}_1^{(3a)} + \hat{V}_{1\eta}^{(3a)} + \mathrm{i}a_0\hat{W}_1^{(3a)} = -\hat{U}_{1\tilde{X}}^{(1)} - \mathrm{i}a_1\hat{W}_1^{(1)}, \tag{4.21 a}$$

$$\begin{aligned}\overline{u}_0\hat{U}_{1\tilde{X}}^{(3a)} &+ \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{U}_1^{(3a)} + \overline{u}_0'\hat{V}_1^{(3a)} \\ &= -[(\overline{u}_0'\eta + \overline{u}_1)\hat{U}_{1\tilde{X}}^{(1)} + \mathrm{i}(\overline{q}_0''\eta^2/2 + \overline{q}_1'\eta + \overline{q}_2)\hat{U}_1^{(1)} + (\overline{u}_0''\eta + \overline{u}_1')\hat{V}_1^{(1)}],\end{aligned} \tag{4.21 b}$$

$$K_0\overline{u}_0\hat{U}_1^{(3a)} + \hat{P}_{1\eta}^{(3a)} = -(K_0\overline{u}_0'\eta + K_1\overline{u}_0 + K_0\overline{u}_1)\hat{U}_1^{(1)}, \tag{4.21 c}$$

$$\begin{aligned}\overline{u}_0\hat{W}_{1\tilde{X}}^{(3a)} &+ \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_1^{(3a)} + \lambda_0\overline{w}_0'\hat{V}_1^{(3a)} + \mathrm{i}a_0\hat{P}_1^{(3a)} \\ &= -[(\overline{u}_0'\eta + \overline{u}_1)\hat{W}_{1\tilde{X}}^{(1)} + \mathrm{i}(\overline{q}_0''\eta^2/2 + \overline{q}_1'\eta + \overline{q}_2)\hat{W}_1^{(1)} \\ &\qquad + (\lambda_0\overline{w}_0''\eta + \lambda_1\overline{w}_0' + \lambda_0\overline{w}_1')\hat{V}_1^{(1)} + \mathrm{i}a_1\hat{P}_1^{(1)}].\end{aligned} \tag{4.21 d}$$

These four equations lead to an equation for $\hat{\Phi}_1^{(3a)}$, namely

$$\hat{N}_{1,1/2-\nu}\hat{\Phi}_1^{(3a)} = \overline{u}_0'\hat{V}_{1\tilde{X}}^{(1)} + \mathrm{i}\overline{q}_0'(r_{11}^{(3a)}\eta + r_{10}^{(3a)})\hat{V}_1^{(1)}, \tag{4.22 a}$$

where

$$r_{10}^{(3a)} = \frac{\overline{q}_1'}{\overline{q}_0'} - \frac{(1+2\nu)}{2}\left(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0}\right) - \frac{a_1}{a_0}(1-2\nu), \tag{4.22 b}$$

$$r_{11}^{(3a)} = \frac{\overline{q}_0''}{\overline{q}_0'} - \frac{(1+2\nu)}{2}\left(\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{\overline{u}_0'}{\overline{u}_0}\right), \tag{4.22 c}$$

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with the solution

$$\hat{\Phi}_1^{(3a)} = \frac{1}{2\nu \mathrm{i}\overline{q}_0'}\left(\overline{u}_0' - \frac{\overline{u}_0 r_{11}^{(3a)}}{(1+2\nu)}\right)\hat{V}_{1\tilde{X}}^{(1)} + \frac{r_{11}^{(3a)}}{(1+2\nu)}\eta\hat{V}_1^{(1)} + \frac{1}{2\nu}\left(r_{10}^{(3a)} - \frac{\overline{q}_1 r_{11}^{(3a)}}{\overline{q}_0'(1+2\nu)}\right)\hat{V}_1^{(1)}. \tag{4.23}$$

(*f*) $O(\epsilon\mu^{3/2-\nu})$ *of the fundamental*

At this order, the governing equations yield

$$\beta\hat{U}_1^{(3b)} + \hat{V}_{1\eta}^{(3b)} + \mathrm{i}a_0\hat{W}_1^{(3b)} = 0, \tag{4.24 a}$$

$$\overline{u}_0\hat{U}_{1\tilde{X}}^{(3b)} + \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{U}_1^{(3b)} + \overline{u}_0'\hat{V}_1^{(3b)} = 0, \tag{4.24 b}$$

$$K_0\overline{u}_0\hat{U}_1^{(3b)} + \hat{P}_{1\eta}^{(3b)} = 0, \tag{4.24 c}$$

$$\overline{u}_0\hat{W}_{1\tilde{X}}^{(3b)} + \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_1^{(3b)} + \lambda_0\overline{w}_0'\hat{V}_1^{(3b)} + \mathrm{i}a_0\hat{P}_1^{(3b)} = 0, \tag{4.24 d}$$

from which it follows that

$$\hat{N}_{1,1/2+\nu}\hat{V}_1^{(3b)} = \tfrac{1}{2}(1-2\nu)\mathrm{i}\overline{q}_0'\,\hat{\Phi}_1^{(3b)} \quad \text{and} \quad \hat{N}_{1,1/2-\nu}\,\hat{\Phi}_1^{(3b)} = 0, \tag{4.25}$$

with solutions

$$\hat{V}_1^{(3b)} = 0 = \hat{\Phi}_1^{(3b)}. \tag{4.26}$$

Thus there is no linear contribution to the evolution equation from inside the critical layer; instead the linear contribution to the evolution equation solely arises from outside the critical layer. Later we shall balance our selected nonlinear term with this order and then derive our second relation involving $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ by matching with the outside asymptotes.

(*g*) $O(\epsilon^2\mu^{2\nu})$ *of the second harmonic*

At this order, the governing equations yield

$$2\beta\hat{U}_2^{(2)} + \hat{V}_{2\eta}^{(2)} + 2\mathrm{i}a_0\hat{W}_2^{(2)} = -\hat{U}_{2\tilde{X}}^{(1)} - 2\mathrm{i}a_1\hat{W}_2^{(1)}, \tag{4.27 a}$$

$$\begin{aligned}
&\overline{u}_0\hat{U}_{2\tilde{X}}^{(2)} + 2\mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{U}_2^{(2)} + \overline{u}_0'\hat{V}_2^{(2)} \\
&\quad = -[(\overline{u}_0'\eta + \overline{u}_1)\hat{U}_{2\tilde{X}}^{(1)} + \mathrm{i}(\overline{q}_0''\eta^2 + 2\overline{q}_1'\eta + 2\overline{q}_2)\hat{U}_2^{(1)} + (\overline{u}_0''\eta + \overline{u}_1')\hat{V}_2^{(1)}] \\
&\qquad -[2\beta\hat{U}_1^{(1)}\hat{U}_1^{(3a)} + \hat{U}_1^{(1)}\hat{U}_{1\tilde{X}}^{(1)} + \hat{V}_1^{(1)}\hat{U}_{1\eta}^{(3a)} + \hat{V}_1^{(3a)}\hat{U}_{1\eta}^{(1)} \\
&\qquad\quad + \mathrm{i}a_0\hat{W}_1^{(1)}\hat{U}_1^{(3a)} + \mathrm{i}a_0\hat{W}_1^{(3a)}\hat{U}_1^{(1)}],
\end{aligned} \tag{4.27 b}$$

$$K_0\overline{u}_0\hat{U}_2^{(2)} + \hat{P}_{2\eta}^{(2)} = -\tfrac{1}{2}K_0\hat{U}_1^{(1)}\hat{U}_1^{(1)} - (K_0\overline{u}_0'\eta + K_1\overline{u}_0 + K_0\overline{u}_1)\hat{U}_2^{(1)}, \tag{4.27 c}$$

$$\begin{aligned}
&\overline{u}_0\hat{W}_{2\tilde{X}}^{(2)} + 2\mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_2^{(2)} + \lambda_0\overline{w}_0'\hat{V}_2^{(2)} + 2\mathrm{i}a_0\hat{P}_2^{(2)} \\
&\quad = -[(\overline{u}_0'\eta + \overline{u}_1)\hat{W}_{2\tilde{X}}^{(1)} + \mathrm{i}(\overline{q}_0''\eta^2 + 2\overline{q}_1'\eta + 2\overline{q}_2)\hat{W}_2^{(1)} \\
&\qquad + (\lambda_0\overline{w}_0''\eta + \lambda_1\overline{w}_0' + \lambda_0\overline{w}_1')\hat{V}_2^{(1)} + 2\mathrm{i}a_1\hat{P}_2^{(1)}] \\
&\qquad -[\beta\hat{U}_1^{(1)}\hat{W}_1^{(3a)} + \beta\hat{U}_1^{(3a)}\hat{W}_1^{(1)} + \hat{U}_1^{(1)}\hat{W}_{1\tilde{X}}^{(1)} + \hat{V}_1^{(1)}\hat{W}_{1\eta}^{(3a)} \\
&\qquad\quad + \hat{V}_1^{(3a)}\hat{W}_{1\eta}^{(1)} + 2\mathrm{i}a_0\hat{W}_1^{(1)}\hat{W}_1^{(3a)}].
\end{aligned} \tag{4.27 d}$$

It is only necessary to determine $\hat{\Phi}_2^{(2)}$ for the analysis at the next order of the fundamental; from the above four equations we find that

$$\begin{aligned}
&\hat{N}_{2,1/2-\nu}\hat{\Phi}_2^{(2)} \\
&\quad = \overline{u}_0'\hat{V}_{2\tilde{X}}^{(1)} + 2\mathrm{i}\overline{q}_0'(r_{11}^{(3a)}\eta + r_{10}^{(3a)})\hat{V}_2^{(1)} + 2(\hat{V}_{1\eta}^{(1)}\hat{\Phi}_{1\eta}^{(3a)} - \hat{V}_1^{(1)}\hat{\Phi}_{1\eta\eta}^{(3a)}) \\
&\qquad - \frac{2\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}(\overline{q}_0'\eta + \overline{q}_1)\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta}^{(1)} + 2\frac{a_1}{a_0}\left(1 - \frac{2\beta\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}\right)(\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta}^{(1)} - \hat{V}_1^{(1)}\hat{V}_{1\eta\eta}^{(1)}) \\
&\qquad + \frac{2\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}(2\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta\tilde{X}}^{(1)} - \hat{V}_{1\tilde{X}}^{(1)}\hat{V}_{1\eta\eta}^{(1)} + \hat{V}_1^{(1)}\hat{V}_{1\eta\eta\tilde{X}}^{(1)}) - \frac{2\overline{u}_0'}{\overline{u}_0}\hat{V}_1^{(1)}\hat{V}_{1\eta}^{(1)}, \qquad (4.28)
\end{aligned}$$

where the constants $r_{10}^{(3a)}$ and $r_{11}^{(3a)}$ are defined by (4.22*b*, *c*). The solution for $\hat{\Phi}_2^{(2)}$ is given in Appendix A.

(*h*) $O(\epsilon^2\mu^{2\nu})$ *of the zeroth harmonic*

At this order, the governing equations are

$$\hat{V}_{0\eta}^{(2)} = 0, \qquad (4.29\,a)$$

$$\begin{aligned}
\overline{u}_0\hat{U}_{0\tilde{X}}^{(2)} + \overline{u}_0'\hat{V}_0^{(2)} &= -(\overline{u}_0'\eta + \overline{u}_1)\hat{U}_{0\tilde{X}}^{(1)} - (\overline{u}_0''\eta + \overline{u}_1')\hat{V}_0^{(1)} - 2(\hat{U}_1^{(1)}\hat{U}_{-1}^{(1)})_{\tilde{X}} \\
&\quad - (\hat{U}_{-1}^{(3a)}\hat{V}_1^{(1)} + \hat{U}_1^{(3a)}\hat{V}_{-1}^{(1)} + \hat{U}_{-1}^{(1)}\hat{V}_1^{(3a)} + \hat{U}_1^{(1)}\hat{V}_{-1}^{(3a)})_\eta \\
&\quad + \frac{4\overline{u}_0'a_1}{(1+2\nu)\mathrm{i}\overline{q}_0'a_0}\left(1 - \frac{2\beta\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}\right)\hat{V}_\eta^{(1)}\hat{V}_{-1\eta}^{(1)}, \qquad (4.29\,b)
\end{aligned}$$

$$K_0\overline{u}_0\hat{U}_0^{(2)} + \hat{P}_{0\eta}^{(2)} = -(K_0\overline{u}_0'\eta + K_0\overline{u}_1 + K_1\overline{u}_0)\hat{U}_0^{(1)} - K_0\hat{U}_1^{(1)}\hat{U}_{-1}^{(1)}, \qquad (4.29\,c)$$

$$\begin{aligned}
\overline{u}_0\hat{W}_{0\tilde{X}}^{(2)} + \lambda_0\overline{w}_0'\hat{V}_0^{(2)} &= -(\overline{u}_0'\eta + \overline{u}_1)\hat{W}_{0\tilde{X}}^{(1)} - (\lambda_0\overline{w}_0''\eta + \lambda_1\overline{w}_0' + \lambda_0\overline{w}_1')\hat{V}_0^{(1)} \\
&\quad - (\hat{U}_1^{(1)}\hat{W}_{-1}^{(1)} + \hat{U}_{-1}^{(1)}\hat{W}_1^{(1)})_{\tilde{X}} \\
&\quad - (\hat{W}_{-1}^{(3a)}\hat{V}_1^{(1)} + \hat{W}_1^{(3a)}\hat{V}_{-1}^{(1)} + \hat{W}_{-1}^{(1)}\hat{V}_1^{(3a)} + \hat{W}_1^{(1)}\hat{V}_{-1}^{(3a)})_\eta \\
&\quad - \frac{2\mathrm{i}a_1}{a_0^2}\left(1 - \frac{2\beta\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}\right)^2\hat{V}_{1\eta}^{(1)}\hat{V}_{-1\eta}^{(1)}. \qquad (4.29\,d)
\end{aligned}$$

In the analysis of the next subsection, at order $(\epsilon^3\mu^{-3/2+3\nu})$ of the fundamental, it is sufficient to know the value of the quantity

$$(\beta - (1+2\nu)\mathrm{i}\overline{q}_0'/2\overline{u}_0')\hat{U}_{0\eta}^{(2)} + \mathrm{i}a_0\hat{W}_{0\eta}^{(2)}, \qquad (4.30)$$

and from (4.29*a*–*d*) it can be shown that

$$\left(\beta - \frac{(1+2\nu)\mathrm{i}\overline{q}_0'}{2\overline{u}_0'}\right)\hat{U}_{0\eta}^{(2)} + \mathrm{i}a_0\hat{W}_{0\eta}^{(2)} = -\frac{1}{\nu(1+2\nu)\mathrm{i}\overline{q}_0'}((r_{01}^{(2)}\eta + r_{00}^{(2)})\hat{V}_{1\eta}^{(1)}\hat{V}_{-1\eta}^{(1)})_{\eta\eta}, \quad (4.31\,a)$$

where

$$r_{00}^{(2)} = r_{10}^{(3a)} - \frac{\overline{u}_0'\overline{q}_1}{\overline{u}_0\overline{q}_0'} + 2\nu\left(\frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0} - \frac{a_1}{a_0}\left(1 - \frac{2\beta\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}\right) - \frac{2}{(1+2\nu)}\frac{\overline{u}_0'\overline{q}_1}{\overline{u}_0'\overline{q}_1}\right), \qquad (4.31\,b)$$

$$r_{01}^{(2)} = r_{11}^{(3a)} + \frac{(4\nu^2 - 2\nu - 1)\overline{u}_0'}{(1+2\nu)\overline{u}_0}. \qquad (4.31\,c)$$

(i) $O(\epsilon^3\mu^{-3/2+3\nu})$ *of the fundamental*

It is at this order that the cubic nonlinear jump due to supercriticality (non-neutrality) effects will arise. The governing equations (4.3*a*–*d*) eventually yield at this order

$$\hat{N}_{1,1/2+\nu}\hat{V}_1^{(4)} = \tfrac{1}{2}(1-2\nu)\mathrm{i}\overline{q}_0'\hat{\Phi}_1^{(4)} + R_1^{(4)}, \qquad \hat{N}_{1,1/2-\nu}\hat{\Phi}_1^{(4)} = R_2^{(4)}. \tag{4.32}$$

It is convenient to write

$$R_2^{(4)} = F_1^{(2)} + R_3^{(4)}, \tag{4.33}$$

where $F_1^{(2)}$ is a function proportional to $\hat{V}_1^{(2)}$ while $R_3^{(4)}$ is given explicitly by

$$\begin{aligned} R_3^{(4)} = {}& \frac{\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}\Bigg[4\hat{V}_2^{(1)}\hat{V}^{(1)}_{-1\eta\eta\tilde{X}} - (\hat{V}^{(1)}_{2\eta}\hat{V}^{(1)}_{-1\tilde{X}})_\eta - \tfrac{1}{2}\hat{V}^{(1)}_{-1}\hat{V}^{(1)}_{2\eta\eta\tilde{X}} + 2\hat{V}^{(1)}_{2\tilde{X}}\hat{V}^{(1)}_{-1\eta\eta} \\ & + \frac{4}{(1+2\nu)\mathrm{i}\overline{q}_0'}(2\hat{V}^{(1)}_{1\eta}(\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_{1\eta})_{\eta\tilde{X}} - (\hat{V}^{(1)}_1(\hat{V}^{(1)}_{1\eta}\hat{V}^{(1)}_{-1\eta})_{\eta\eta})_{\tilde{X}} + \hat{V}^{(1)}_{1\eta\tilde{X}}(\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_{1\eta})_\eta)\Bigg] \\ & + \frac{2}{(1+2\nu)\mathrm{i}\overline{q}_0'}\left(\frac{\overline{u}_0'}{\overline{u}_0}\eta + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0} - \frac{a_1}{a_0}(1 - 2\beta\overline{u}_0'/(1+2\nu)\mathrm{i}\overline{q}_0')\right)\hat{V}^{(1)}_1(\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_{1\eta})_{\eta\eta} \\ & + \frac{2\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'\overline{u}_0}\Bigg[\mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\left(\frac{2}{(1+2\nu)\mathrm{i}\overline{q}_0'}\hat{V}^{(1)}_{1\eta}(\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_{1\eta})_\eta - \tfrac{1}{2}\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_{2\eta}\right) \\ & \qquad + \hat{V}^{(1)}_{1\eta}(\hat{V}^{(1)}_{-1}\hat{V}^{(1)}_{1\eta})_\eta\Bigg] \\ & + \hat{V}^{(1)}_2\hat{\Phi}^{(3a)}_{-1\eta\eta} - \hat{\Phi}^{(3a)}_{1\eta}(\beta\hat{U}^{(1)}_0 + \mathrm{i}a\hat{W}^{(1)}_0) \\ & + \hat{V}^{(1)}_1\{(\beta - (1+2\nu)\mathrm{i}\overline{q}_0'/2\overline{u}_0')\hat{U}^{(2)}_{0\eta} + \mathrm{i}a\hat{W}^{(2)}_{0\eta}\} \\ & - \frac{\overline{u}_0'}{2\overline{u}_0}(\hat{V}^{(1)}_{2\eta}\hat{V}^{(1)}_{-1} + 2\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_2) + \tfrac{1}{2}\hat{V}^{(1)}_{2\eta}\hat{\Phi}^{(3a)}_{-1\eta} \\ & - \hat{V}^{(1)}_{-1\eta}\hat{\Phi}^{(2)}_{2\eta} - \tfrac{1}{2}\hat{V}^{(1)}_{-1}\hat{\Phi}^{(2)}_{2\eta\eta}. \end{aligned} \tag{4.34}$$

To derive an evolution equation the asymptotic representation of $\hat{V}_1^{(4)}$ is needed. This is

$$\hat{V}_1^{(4)} = C_\pm^{(4)}\eta^{1/2+\nu} + D_\pm^{(4)}\eta^{1/2-\nu} + O(\eta^{-1/2+\nu}) \quad \text{as} \quad \eta \to \pm\infty, \tag{4.35}$$

where, in particular, $D_+^{(4)} - D_-^{(4)}$ is given by

$$\begin{aligned} D_+^{(4)} - D_-^{(4)} = {}& -\frac{\mathrm{i}^{1/2-\nu}}{2\nu\overline{u}_0\Gamma(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1/2-\nu} \\ & \times\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^\infty \mathrm{d}\tilde{x}_4\tilde{x}_4^{1/2-\nu}R_3^{(4)}(\tilde{X}-\tilde{x}_4,\eta)\exp\{-\mathrm{i}\tilde{x}_4(\overline{q}_0'\eta + \overline{q}_1)/\overline{u}_0\}. \end{aligned} \tag{4.36}$$

It is convenient to split $R_3^{(4)}$ into five parts:

$$\begin{aligned} D_+^{(4)} - D_-^{(4)} = {}& (D_+^{(4)} - D_-^{(4)})_{00} + (D_+^{(4)} - D_-^{(4)})_{01} + (D_+^{(4)} - D_-^{(4)})_{20} \\ & + (D_+^{(4)} - D_-^{(4)})_{21} + (D_+^{(4)} - D_-^{(4)})_{22}, \end{aligned} \tag{4.37}$$

where the terms on the right-hand side are defined in Appendix B. Matching the

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inner asymptote, (4.35), with the outer asymptote, (3.17*a*), leads to the relations,

$$D_{+}^{(4)} = b_{1+}^{(2)} B_{+}, \qquad D_{-}^{(4)} = \mathrm{i}^{1-2\nu} b_{1-}^{(2)} B_{-}, \tag{4.38}$$

which can be combined to give

$$D_{+}^{(4)} - D_{-}^{(4)} = B(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)}); \tag{4.39}$$

thus a second relation between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ has been determined (the first being given by the solvability condition (3.14)).

5. The cubic-nonlinearity due to viscosity

With the aim to derive an evolution equation with cubic-nonlinearity directly due to viscous effects, it is necessary to consider some additional terms in the critical layer analysis; in fact, instead of the expansions (4.6), it is necessary to consider

$$\begin{aligned}\hat{V}_1 = {} & \epsilon\mu^{1/2+\nu}\hat{V}_1^{(1)} + \ldots + \epsilon^3\mu^{-5/2+3\nu}\hat{V}_1^{(2)} + \ldots + \epsilon\mu^{3/2+\nu}\hat{V}_1^{(3a)} + \ldots \\ & + \epsilon\mu^{3/2-\nu}\hat{V}_1^{(3b)} + \ldots + \epsilon^3\mu^{-3/2+3\nu}\hat{V}_1^{(2)} + \ldots \\ & + \epsilon G^{-1/2}\mu^{-5/2+\nu}\hat{V}_1^{(5)} + \ldots + \epsilon^3 G^{-1/2}\mu^{-11/2+3\nu}\hat{V}_1^{(6)} + \ldots, \end{aligned} \tag{5.1 a}$$

$$\hat{V}_0 = \epsilon^2\mu^{-1+2\nu}\hat{V}_0^{(1)} + \ldots + \epsilon^2\mu^{2\nu}\hat{V}_0^{(2)} + \ldots + \epsilon^2 G^{-1/2}\mu^{-4+2\nu}\hat{V}_0^{(3)} + \ldots, \tag{5.1 b}$$

$$\hat{V}_2 = \epsilon^2\mu^{-1+2\nu}\hat{V}_2^{(1)} + \ldots + \epsilon^2\mu^{2\nu}\hat{V}_2^{(2)} + \ldots + \epsilon^2 G^{-1/2}\mu^{-4+2\nu}\hat{V}_2^{(3)} + \ldots, \tag{5.1 c}$$

with similar expansions for $\hat{U}$, $\hat{W}$, $\hat{P}$ and $\hat{\Phi}$.

One would expect to obtain this 'viscous' cubic nonlinearity from balancing the nonlinear fundamental term occurring at $O(\epsilon^3 G^{-1/2}\mu^{-11/2+3\nu})$ with the fundamental term occurring at $O(\epsilon\mu^{3/2-\nu})$ and then matching with the solutions outside of the critical-layer to provide a second relationship between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$. However, for the problem being considered here (namely, that of inviscid Görtler vortices in an *in*compressible three-dimensional boundary-layer) this proves fruitless. It is illustrative to consider the governing equations for the fundamental in the critical layer, at $O(\epsilon G^{-1/2}\mu^{-5/2+\nu})$, where viscous terms first appear on the right-hand sides. These are

$$\beta\hat{U}_1^{(5)} + \hat{V}_{1\eta}^{(5)} + \mathrm{i}a_0\hat{W}_1^{(5)} = 0, \tag{5.2 a}$$

$$\overline{u}_0\hat{U}_{1\tilde{X}}^{(5)} + \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{U}_1^{(5)} + \overline{u}_0'\hat{V}_1^{(5)} = \hat{U}_{1\eta\eta}^{(1)}, \tag{5.2 b}$$

$$K_0\overline{u}_0\hat{U}_1^{(5)} + \hat{P}_{1\eta}^{(5)} = 0, \tag{5.2 c}$$

$$\overline{u}_0\hat{W}_{1\tilde{X}}^{(5)} + \mathrm{i}(\overline{q}_0'\eta + \overline{q}_1)\hat{W}_1^{(5)} + \lambda_0\overline{w}_0'\hat{V}_1^{(5)} + \mathrm{i}a_0\hat{P}_1^{(5)} = \hat{W}_{1\eta\eta}^{(1)}, \tag{5.2 d}$$

which, in particular, lead to the equation

$$\hat{N}_{1,1/2-\nu}\hat{\Phi}_1^{(5)} = 0, \tag{5.3}$$

with solution

$$\hat{\Phi}_1^{(5)} = 0. \tag{5.4}$$

Basically, this result is due to the symmetry of the governing equations (5.2*a*–*d*) at this order; in particular, is is essentially due to the fact that the 'viscous' terms on the right-hand sides of (5.2*b*, *d*) both have the same coefficient (namely 'one'). This symmetry can be broken in the associated problem for stratified shear-flows by a choice of a non-unity Prandtl number (see Churilov & Shukhman 1988; Dando 1996). However, since here we are considering an incompressible flow, the energy equation does not enter into the problem and hence neither does the Prandtl number, i.e. whatever the value of the Prandtl number of the flow, $\hat{\varPhi}_1^{(5)}$ is still zero since the Prandtl number cannot break the symmetry of equations (5.2*a*–*d*).

Thus, there is no evolution equation, with cubic-nonlinearity directly due to viscous effects, for this problem. Note, however, that viscosity (actually manifested here by the Görtler number G; since the inviscid limit corresponds to $G \gg 1$) still plays an important role in this study since it's value will determine/limit the range of applicability/validity of our final evolution equations. In fact, if the supercriticality of the disturbance falls below a certain size (namely $\mu < G^{-1/6}$), then the critical layer is dominated by viscous effects and the non-equilibrium critical-layer theory employed here is not (directly) relevant. Also, the magnitude of viscous effects (as characterized/measured by the value of G relative to μ) determines whether the linear term in the evolution equations is proportional to $\tilde{X}$ or not.

6. The evolution equation

Recalling that the cubic-nonlinearity directly due to viscous effects does not arise for the problem under consideration, the evolution equation can be written in 'composite' form

$$(I_3 - \mathrm{i}^{-4\nu} I_1)\frac{\partial B}{\partial \tilde{X}} + (I_4 - \mathrm{i}^{-4\nu} I_2)B = 2\nu B\{\epsilon^2\mu^{-3+4\nu}(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)})_{sc} + \epsilon^4\mu^{-7+6\nu}(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)})_q\}, \quad (6.1)$$

where $(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)})_{sc}$ denotes the nonlinear part of the jump due to supercriticality effects (calculated in § 4), while $(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)})_q$ denotes the total part of the jump due to the quintic nonlinearity. An expression for $(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)})_q$ can be derived in a similar manner to that in which the corresponding term was calculated in the stratified-shear layer studies of Churilov & Shukhman (1988) and Blackaby *et al.* (1993). However, since this would be an extremely long and complicated task for the present problem, for the moment instead we shall concentrate on obtaining some numerical results for the cubic evolution equation (which will also determine how relevant the quintic nonlinearity is). Note that the evolution equation (6.1), and other forms of it appearing later in this section, is valid provided $\mu \gg G^{-1/4}$; for smaller values of supercriticality, the non-parallelism of the underlying flow is significant and the linear term will include the additional factor $\tilde{X}$ (see, for instance, the amplitude equations (4.1*a*, *b*) in the paper of Hall & Smith (1984)).

In figure 4 we show the regions of influence of the nonlinear terms in the amplitude equation (6.1). The cubic-nonlinearity due to supercriticality effects becomes important when the amplitude of the disturbance $\epsilon = \mu^{3/2-2\nu}$ (when the cubic term is as large as the linear terms in the evolution equation, 6.1). The quintic term is of the same size as the cubic term when $\epsilon = \mu^{2-\nu}$ ($\gg \mu^{3/2-2\nu}$; since μ is small and $\nu < -1/2$ for the neutral, inviscid Görtler modes). Thus, if the evolution equation just containing the cubic nonlinearity leads to a sufficient increase in the amplitude of the disturbance, the quintic nonlinear term will become the dominant term in the 'composite' evolution equation (6.1).

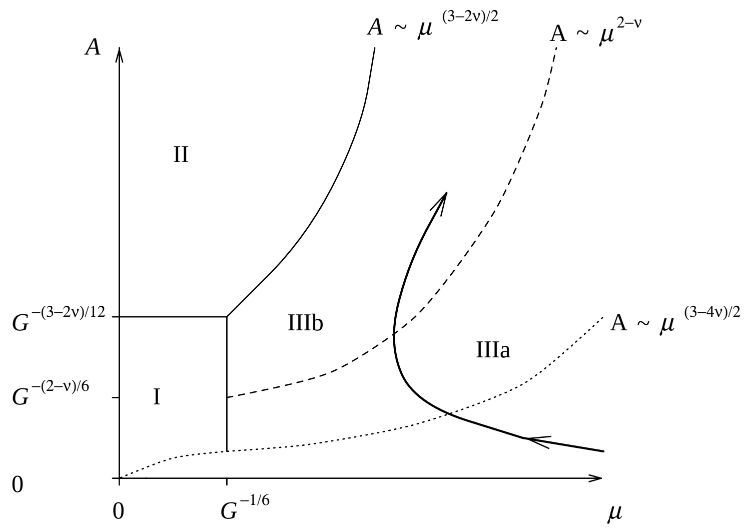

Figure 4. A diagram of the various regimes of the critical layer. I, viscous, steady critical layer; Landau–Stuart–Watson theory; II, strongly nonlinear, equilibrium critical layer; Benney & Bergeron theory. IIIa, unsteady critical layer; largest nonlinear term in (6.1) is the cubic one due to supercriticality effects; IIIb, unsteady critical layer; largest nonlinear term in (6.1) is the quintic one. Solid lines represent boundaries with areas where the critical layers are not unsteady, dashed lines represent boundaries between different base evolution equations from (6.1) and the dotted lines indicate the threshold values at which the nonlinear evolution equations become valid. The thick line on the diagram indicates the predicted evolutionary path of the disturbance for small values of the crossflow parameter.

(*a*) *Numerical results for the equation with cubic nonlinearity*

Let us now concentrate on obtaining numerical results for the amplitude equation with only the the cubic-nonlinearity present. To ease numerical calculations, the jump expression $D_+^{(4)} - D_-^{(4)}$ is rewritten in so-called kernel form

$$D_+^{(4)} - D_-^{(4)} = \frac{\mathrm{i}^{1-2\nu}(1+2\nu)^2}{4\nu\,\overline{u}_0^2\Gamma^2(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}\int_0^\infty \mathrm{d}s\, s^{1-4\nu}\int_0^1 \mathrm{d}\sigma$$
$$\times\{B(\tilde{X}-s)B(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)(sG_1(\sigma)+G_2(\sigma)$$
$$+B_{\tilde{X}}(\tilde{X}-s)B(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)sG_3(\sigma)$$
$$+B(\tilde{X}-s)B_{\tilde{X}}(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)s\,G_4(\sigma)$$
$$+B(\tilde{X}-s)\,B(\tilde{X}-\sigma s)\overline{B}_{\tilde{X}}(\tilde{X}-(1+\sigma)s)s\,G_5(\sigma)\}, \qquad (6.2)$$

where the kernel functions $G_1(\sigma),\ldots,G_5(\sigma)$ are defined in Appendix C. An inspection of this kernel representation shows that things are considerably more complicated than for the corresponding temporal, stratified-shear flow problem. The nonlinear-jump is complex (i.e. it has real and imaginary parts) and there are cubic nonlinear terms containing $\tilde{X}$-derivatives. The occurrence of spatial derivatives of the amplitude function in the nonlinear terms of the evolution equation is a relatively novel feature for non-equilibrium critical-layer studies. Previously, such streamwise derivatives have only been seen in the study of Churilov & Shukhman (1994), concerning the spatial evolution of helical disturbances to an axial jet; while spanwise derivatives of the amplitude function in the nonlinear term arise in the evolution equations derived by Wu (1993), concerning the temporal-spatial modulation of near-planar Rayleigh

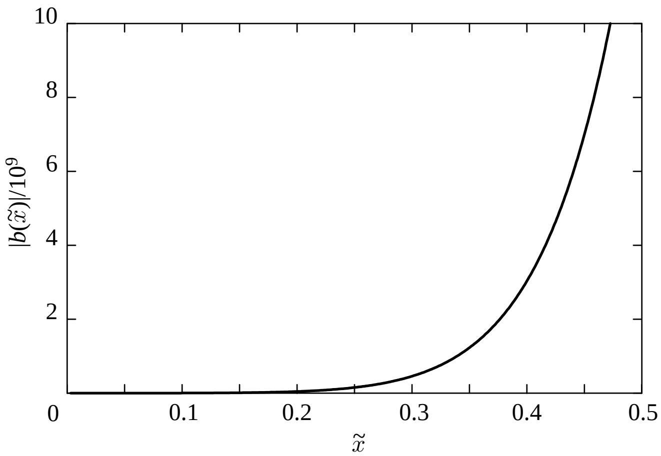

Figure 5. The amplitude of $b(\tilde{x})$ for $\lambda_0 = 25$.

waves in shear flows, and Gajjar (1995), concerning stationary crossflow vortices in fully three-dimensional boundary layers. As with the work of Churilov & Shukhman (1994), the inclusion of these nonlinear spatial derivative terms is wholly associated with the spatial formulation of the problem; note that there were no equivalent terms in the related temporal stability study for stratified shear layers.

Formally setting $\epsilon = \mu^{3/2-2\nu}$ and using the relations (4.39), (6.2) to substitute for $(b_{1+}^{(2)} - \mathrm{i}^{-4\nu} b_{1-}^{(2)})_{sc}$ in equation (6.1), leads to the evolution equation

$$\begin{aligned}\frac{\partial B}{\partial \tilde{X}} = \gamma_1 B + \gamma_2 \int_0^\infty \mathrm{d}s\, s^{1-4\nu} \int_0^1 \mathrm{d}\sigma \\ \times\{B(\tilde{X}-s)B(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)(sG_1(\sigma)+G_2(\sigma)) \\ +B_{\tilde{X}}(\tilde{X}-s)B(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)sG_3(\sigma) \\ +B(\tilde{X}-s)B_{\tilde{X}}(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)sG_4(\sigma) \\ +B(\tilde{X}-s)B(\tilde{X}-\sigma s)\overline{B}_{\tilde{X}}(\tilde{X}-(1+\sigma)s)sG_5(\sigma)\},\end{aligned} \tag{6.3}$$

where

$$\gamma_1 = \frac{(\mathrm{i}^{-4\nu} I_2 - I_4)}{(I_3 - \mathrm{i}^{-4\nu} I_1)}, \quad \gamma_2 = \frac{\mathrm{i}^{1-2\nu}(1+2\nu)^2}{(I_3 - \mathrm{i}^{-4\nu} I_1) 2\overline{u}_0^2 \Gamma^2(\frac{1}{2}-\nu)} \left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}, \tag{6.4}$$

with the initial condition $B(\tilde{X}) \to 0$ as $\tilde{X} \to -\infty$. This evolution equation is the main analytical result of our study; note that it is an integro-differential equation whose kernel contains nonlinear 'cubic' combinations of the amplitude function B and its derivative. Moreover, note that the nonlinear term depends on the entire history of the evolution, rather than just on the value of B and $B_{\tilde{X}}$ at the location $\tilde{X}$. Such evolution equations (though generally not containing derivatives in the nonlinear terms) often arise from non-equilibrium critical-layer studies; the study by Hickernell (1984) is usually cited as the first in which the weakly nonlinear analysis leads to an integro-differential equation rather than the more familiar Stuart–Landau–Watson equation.

We consider the solution properties of (6.3) both analytically and numerically. In Appendix D we consider analytically the possibility that some solutions of (6.3)

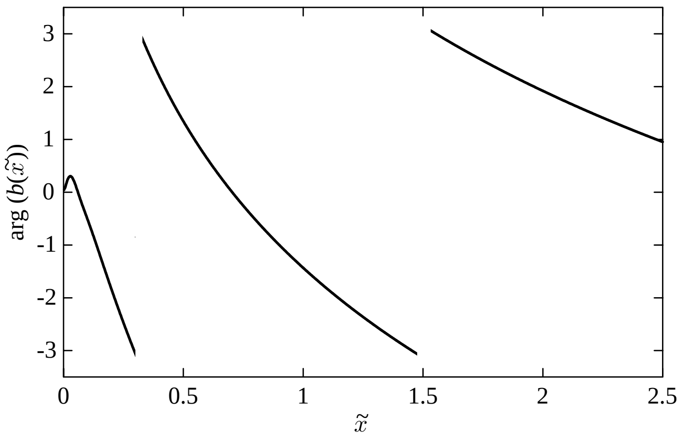

Figure 6. The argument of $b(\tilde{x})$ (in the range $-\pi$ to $+\pi$) for $\lambda_0 = 37.5$.

will terminate abruptly in an algebraic singularity. Let us now consider numerical calculations for which it is convenient to introduce the variable

$$\tilde{x} = |B_0|^2 \exp\{2\gamma_{1r}\tilde{X}\}, \tag{6.5}$$

with γ_{1r} denoting the real part of γ_1 (this is similar to the 'logarithmic time' variable which was introduced by Churilov & Shukhman 1988). We also set

$$B(\tilde{X}) = B_0 b(\tilde{X}) \exp\{\gamma_1 \tilde{X}\}, \tag{6.6}$$

where the constant B_0 is chosen such that $b(\tilde{X}) \to 1$ as $\tilde{X} \to -\infty$, note that the requirement $B(\tilde{X}) \to 0$ as $\tilde{X} \to -\infty$ is satisfied as our numerical calculations confirm γ_{1r} to be always positive. After this we find that the evolution equation reduces to

$$\begin{aligned}\frac{\partial b}{\partial \tilde{x}} = \int_0^1 \mathrm{d}\sigma \int_0^\infty \mathrm{d}r\, r^{1-4\nu} e^{-2\gamma_{1r}r} \Big\{ & b(\tilde{x}e^{-r/(1+\sigma)}) b(\tilde{x}e^{-\sigma r/(1+\sigma)}) \bar{b}(\tilde{x}e^{-r})(rG_A + G_B) \\ & + \frac{\partial b}{\partial \tilde{x}}(\tilde{x}e^{-r/(1+\sigma)}) b(\tilde{x}e^{-\sigma r/(1+\sigma)}) \bar{b}(\tilde{x}e^{-r}) \tilde{x} r G_C \\ & + b(\tilde{x}e^{-r/(1+\sigma)}) \frac{\partial b}{\partial \tilde{x}}(\tilde{x}e^{-\sigma r/(1+\sigma)}) \bar{b}(\tilde{x}e^{-r}) \tilde{x} r G_D \\ & + b(\tilde{x}e^{-r/(1+\sigma)}) b(\tilde{x}e^{-\sigma r/(1+\sigma)}) \frac{\partial \bar{b}}{\partial \tilde{x}}(\tilde{x}e^{-r}) \tilde{x} r G_E \Big\},\end{aligned} \tag{6.7}$$

where

$$G_A(\sigma) = \frac{\gamma_2}{2\gamma_{1r}(1+\sigma)^{2-4\nu}} (G_1(\sigma) + G_3(\sigma) + G_4(\sigma) + G_5(\sigma)), \tag{6.8 a}$$

$$\{G_B(\sigma), G_C(\sigma), G_D(\sigma), G_E(\sigma)\} = \frac{\gamma_2}{2\gamma_{1r}(1+\sigma)^{1-4\nu}} \{G_2(\sigma), G_3(\sigma), G_4(\sigma), G_5(\sigma)\}. \tag{6.8 b}$$

Let us now consider some numerical solutions of the evolution equation (6.7). It is not possible to eliminate the parameter x_1 from the equation as it does not occur in all terms (cf. Blackaby *et al.* (1993) where the parameter J_1 was eliminated from the final evolution equation) and so for these numerical solutions we have taken the representative value $x_1 = -1.0$ (note that we are considering a location upstream

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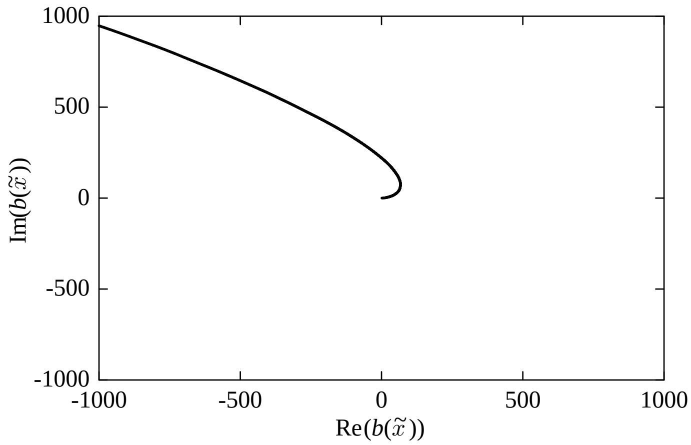

Figure 7. The real (x-axis) and imaginary (y-axis) parts of $b(\tilde{x})$, showing its evolution as $\tilde{x}$ increases for $\lambda_0 = 55$.

of the neutral location). In order to focus attention on the effect of increasing $\hat{a}$ (a consequence of boundary-layer growth and the experimental observation that Görtler vortices conserve their physical spanwise wavenumber) we choose to consider a crossflow which does not vary with streamwise location and so set $\lambda_1 = 0$. In figure 5 we show the magnitude of $b(\tilde{x})$ for the case when $\lambda_0 = 25.0$, this is close to the neutral point $\hat{a} = 1.316$. It is easily seen that a large increase in amplitude occurs very rapidly and we suspect that the solution is breaking up. In Appendix D we consider algebraic singularity type solutions of (6.3), (6.7) but are unable to prove their existence. We find that $b(\tilde{x})$ actually evolves in a spiral (in the complex plane of b) and in an effort to show this we have plotted in figure 6 the argument of $b(\tilde{x})$ (in the range $-\pi$ to $+\pi$) for the case when $\lambda_0 = 37.5$. We find that as the value of the crossflow is increased the growth of the disturbance is damped and does not occur as rapidly. In figure 7 we have plotted the real part of $b(\tilde{x})$ on the x-axis and the imaginary part on the y-axis for a crossflow value of $\lambda_0 = 55.0$. This plot covers the evolution of $b(\tilde{x})$ from $\tilde{x} = 0$ through to $\tilde{x} = 4.767$ and although this amplitude growth is considerably less than that for the case when $\lambda_0 = 25.0$ (figure 5) it is still large enough to make the spiral pattern difficult to present graphically. We predict that the rapid increase in the disturbance amplitude for these three cases means that the disturbance will enter region IIIb in figure 4 where the dominant nonlinear term in the evolution equation is the quintic. This means that it may be necessary in the future to derive the quintic nonlinear term via a critical layer analysis similar to that of § 4 and obtain numerical solutions for this new evolution equation. However, if we increase the crossflow further we find that this rapid increase in the disturbance amplitude does not occur. Considering a value of $\lambda_0 = 75.0$ in figure 8 we can see that there is hardly any increase in the disturbance amplitude. Consequently, it is unlikely that the disturbance will enter region IIIb in figure 4 where the evolution equation would be dominated by the quintic nonlinearity.

We have shown that, at least for the nonlinear evolution equation with cubic nonlinearity due to supercriticality effects, crossflow has the same effect as with a linear stability study: it has a stabilizing influence on inviscid Görtler vortices. This stabilizing influence manifests itself in delaying the large amplitude growth of the disturbance. The large amplitude growth which occurs for smaller values of

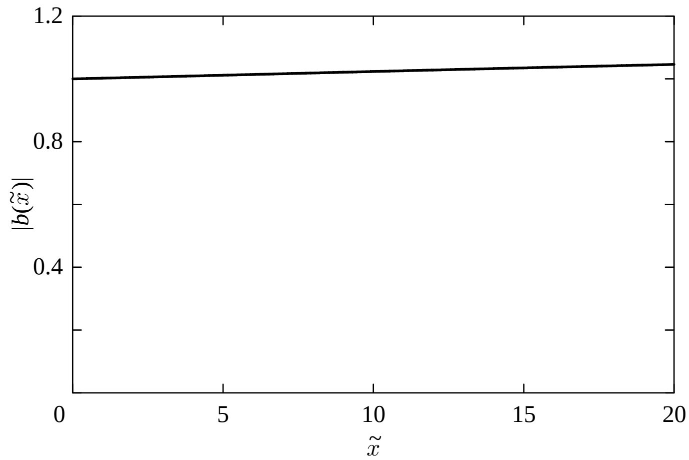

Figure 8. The amplitude of $b(\tilde{x})$ for $\lambda_0 = 75$.

the crossflow suggests that in future work it is necessary to consider the evolution equation with quintic nonlinearity. The much smaller amplitude growth which occurs for larger values of the crossflow necessitates that we will have to consider a steady, viscous critical-layer analysis for the subsequent evolution of these modes. Recall that viscous-spreading of the boundary-layer results in the growth rate decreasing and so in the absence of large-amplitude growth (which only occurs for smaller crossflow values) the disturbance will enter region I in figure 4.

7. Conclusion

In this paper we have considered the nonlinear development of inviscid Görtler vortices in an incompressible boundary-layer flow using non-equilibrium critical-layer theory. We knew from earlier work concerned with stratified-shear flows that it was necessary to consider three different possible integro-differential amplitude equations; one where the nonlinearity is cubic in nature and is directly due to viscous effects; another where the nonlinearity is also cubic in nature but due to supercriticality (non-neutrality) effects; and a third where the nonlinearity is quintic in nature. In this paper, attention has been concentrated on the former two since the two evolution equations with cubic-nonlinearities are relevant for smaller disturbance sizes. However, in §5 we demonstrated that for the incompressible flow being considered here, the nonlinear cubic-jump directly due to viscous effects is zero and hence there is no associated 'viscous-cubic' evolution equation. Even if this was not the case, we note that based on our assumptions concerning viscous-spreading resulting in an unstable linear disturbance mode approaching a later neutral state, initially the disturbance is associated with locations in the bottom right-hand corner of figure 4, indicating that the evolution equation with cubic nonlinearity due to supercriticality (non-neutrality) effects merits the first attention. Consequently, in this paper we have concentrated on obtaining numerical results for this evolution equation. In fact, our spatial-stability approach has resulted in streamwise derivatives of the amplitude function appearing in the nonlinear term; this appears to be a relatively novel feature of our integro-differential evolution equation.

The numerical solutions show that for small values of the crossflow parameter the disturbance amplitude evolves by describing a spiral, in the complex plane, of rapidly increasing amplitude. This large increase in the amplitude of the disturbance means that the evolution process will soon move on to a stage where the evolution of the mode is governed by an integro-differential amplitude equation with a quintic nonlinearity. In the related study of Blackaby *et al.* (1993) the integro-differential equation with a quintic nonlinearity leads to continued growth of the disturbance amplitude which results in the effects of nonlinearity spreading to outside the critical level, by which time the flow has become fully nonlinear. Although we have noted many similarities between of the nonlinear evolution of modes on unstable stratified shear layers, and the nonlinear evolution of inviscid Görtler vortices in three-dimensional boundary layers, we are unable to deduce the properties of solutions of the quintic integro-differential equation for the Görtler problem from the earlier results of Blackaby *et al.* (1993) because of the major differences in the kernel functions for the two problems. The daunting task of deriving, and obtaining numerical solutions of, the integro-differential with quintic nonlinearity for the Görtler problem will need to be undertaken in order to clarify further the nonlinear evolution process of inviscid Görtler vortices in flows that are only very slightly three dimensional. However, for larger values of the crossflow we find that the disturbance amplitude still evolves by describing a spiral in the complex plane but its amplitude increases only very slowly, indicating that crossflow has a significant stabilizing influence, on the Görtler instability, for the nonlinear problem as well as the linear problem.

As mentioned above, further work is needed to address the subsequent nonlinear evolution of the disturbance once its size has grown such that it is governed by the integro-differential equation with quintic nonlinearity. Another extension of the work in this paper would be to consider compressible three-dimensional boundary-layer flows; linear stability studies of such flows show that a larger crossflow is needed to stabilize the Görtler vortices and it would be interesting to see what effect compressibility has on the results presented here. Finally, we note that the theory developed here can be applied to other flows (e.g. it could be used to study of the nonlinear evolution of inviscid vortex instabilities in the three-dimensional flow above a heated plate).

The authors are extremely grateful to Professor S. M. Churilov for his detailed correspondence regarding Churilov & Shukhman (1988). The second author is grateful to colleagues at Imperial College who gave him advice and encouragement regarding the numerical computations when he presented a seminar there. The authors are also grateful to the referees for their helpful comments on the original manuscript.

Appendix A. The solution of equation (4.28)

On writing

$$\Delta_1 = \frac{1}{2\nu}\left(r_{10}^{(3a)} - \frac{\overline{q}_1 r_{11}^{(3a)}}{(1+2\nu)\overline{q}_0'}\right), \tag{A 1 a}$$

$$\Delta_2 = \frac{a_1}{a_0}\left(1 - \frac{2\beta\overline{u}_0'}{(1+2\nu)\mathrm{i}\overline{q}_0'}\right), \tag{A 1 b}$$

$$\Delta_3 = \frac{\overline{u}_0'\overline{q}_1}{(1+2\nu)\overline{u}_0\overline{q}_0'}, \tag{A 1 c}$$

where $r_{10}^{(3a)}$ and $r_{11}^{(3a)}$ are defined by $(4.22a, b)$, the solution of equation (4.28) can be written

$$
\begin{aligned}
\hat{\Phi}_2^{(2)} = & -\frac{\mathrm{i}^{-1-2\nu}(1+2\nu)^2}{4\overline{u}_0^2\Gamma^2(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-1-2\nu} \\
& \times\int_0^\infty \mathrm{d}\overline{x}\int_0^\infty \mathrm{d}\hat{x}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\,(x_1x_2)^{-3/2-\nu} \\
& \times(x_1-x_2)^2(x_1+x_2)^{1/2+\nu}(2\hat{x}+x_1+x_2)^{-1-2\nu} \\
& \times(2\overline{x}+2\hat{x}+x_1+x_2)^{-3/2+\nu} \\
& \times[(\overline{u}_0\,\partial/\partial\tilde{X}+2\mathrm{i}\overline{q}_0'[r_{11}^{(3a)}\eta+r_{10}^{(3a)}]) \\
& \times(B(\tilde{X}-\overline{x}-\hat{x}-x_1)B(\tilde{X}-\overline{x}-\hat{x}-x_2))] \\
& \times\exp\{-\mathrm{i}(2\overline{x}+2\hat{x}+x_1+x_2)(\overline{q}_0'\eta+\overline{q}_1)/\overline{u}_0\} \\
& -\frac{\mathrm{i}^{-1-2\nu}(1+2\nu)}{4\overline{u}_0^2\Gamma^2(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-1-2\nu} \\
& \times\int_0^\infty \mathrm{d}\overline{x}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\,(x_1x_2)^{-3/2-\nu} \\
& \times(x_1+x_2)^{1/2-\nu}(2\overline{x}+x_1+x_2)^{-3/2+\nu} \\
& \times\Big\{[(x_1+x_2)((1+2\nu)\overline{u}_0'+\overline{u}_0r_{11}^{(3a)}) \\
& \quad -\mathrm{i}\overline{q}_0'(x_1^2+x_2^2)(r_{11}^{(3a)}\eta+(1+2\nu)(\Delta_1+\Delta_2)) \\
& \quad +2\mathrm{i}\overline{q}_0'x_1x_2((2+\overline{u}_0r_{11}^{(3a)}/\overline{u}_0')\eta+(1+2\nu)(\Delta_1+\Delta_2+2\Delta_3))] \\
& \quad \times B(\tilde{X}-\overline{x}-x_1)B(\tilde{X}-\overline{x}-x_2) \\
& \quad +\frac{[(1+6\nu)\overline{u}_0'-\overline{u}_0r_{11}^{(3a)}]}{2\nu}x_1x_2(B(\tilde{X}-\overline{x}-x_1)B(\tilde{X}-\overline{x}-x_2))_{\tilde{X}} \\
& \quad -\overline{u}_0'[x_2^2B_{\tilde{X}}(\tilde{X}-\overline{x}-x_1)B(\tilde{X}-\overline{x}-x_2) \\
& \quad +x_1^2B(\tilde{X}-\overline{x}-x_1)B_{\tilde{X}}(\tilde{X}-\overline{x}-x_2)] \\
& \quad +\frac{[\overline{u}_0r_{11}^{(3a)}-\overline{u}_0']}{2\nu}[x_2^2B(\tilde{X}-\overline{x}-x_1)B_{\tilde{X}}(\tilde{X}-\overline{x}-x_2) \\
& \quad +x_1^2B_{\tilde{X}}(\tilde{X}-\overline{x}-x_1)B(\tilde{X}-\overline{x}-x_2)]\Big\} \\
& \times\exp\{-\mathrm{i}(2\overline{x}+x_1+x_2)(\overline{q}_0'\eta+\overline{q}_1)/\overline{u}_0\}. \qquad (\mathrm{A}\,2)
\end{aligned}
$$

Appendix B. The cubic-nonlinear jump terms

The five parts of the nonlinear cubic jump due to supercriticality effects (see equation (4.38)) are given by

$$
\begin{aligned}
&(D_+^{(4)} - D_-^{(4)})_{00}\\
&= -\frac{\mathrm{i}^{-2\nu}(1+2\nu)\overline{u}_0'}{8\nu\overline{u}_0^2\overline{q}_0'\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3\,(x_1x_2x_3)^{-3/2-\nu}\\
&\quad\times(x_3-x_1-x_2)^{1/2-\nu}\Big\{x_3\overline{B}(\tilde{X}+x_1+x_2-2x_3)\\
&\quad\times[B_{\tilde{X}}(\tilde{X}+x_1-x_3)B(\tilde{X}+x_2-x_3)\,x_1\,(x_1-x_3)^2\\
&\quad+B(\tilde{X}+x_1-x_3)B_{\tilde{X}}(\tilde{X}+x_2-x_3)\,x_2\,(x_2-x_3)^2]\\
&\quad-\frac{1}{4\nu\overline{u}_0'}((1+6\nu)\overline{u}_0'-\overline{u}_0 r_{11}^{(3a)})x_1\,x_2\,x_3\overline{B}(\tilde{X}+x_1+x_2-2x_3)\\
&\quad\times[B_{\tilde{X}}(\tilde{X}+x_1-x_3)B(\tilde{X}+x_2-x_3)(x_1-x_3)\\
&\quad+B(\tilde{X}+x_1-x_3)B_{\tilde{X}}(\tilde{X}+x_2-x_3)(x_2-x_3)]\\
&\quad+B(\tilde{X}+x_1-x_3)B(\tilde{X}+x_2-x_3)\overline{B}_{\tilde{X}}(\tilde{X}+x_1+x_2-2x_3)\\
&\quad\times x_3[x_1(x_1-x_3)^2+x_2(x_2-x_3)^2]\\
&\quad+\overline{B}(\tilde{X}+x_1+x_2-2x_3)x_3\\
&\quad\times[B_{\tilde{X}}(\tilde{X}+x_1-x_3)B(\tilde{X}+x_2-x_3)x_2(x_2-x_3)^2\\
&\quad+B(\tilde{X}+x_1-x_3)B_{\tilde{X}}(\tilde{X}+x_2-x_3)x_1(x_1-x_3)^2]\\
&\quad+2x_1x_2x_3(B(\tilde{X}+x_1-x_3)B(\tilde{X}+x_2-x_3)\\
&\quad\times\overline{B}_{\tilde{X}}(\tilde{X}+x_1+x_2-2x_3)(x_1+x_2-2x_3)\\
&\quad+\overline{B}(\tilde{X}+x_1+x_2-2x_3)[B_{\tilde{X}}(\tilde{X}+x_1-x_3)B(\tilde{X}+x_2-x_3)(x_2-x_3)\\
&\quad+B(\tilde{X}+x_1-x_3)B_{\tilde{X}}(\tilde{X}+x_2-x_3)(x_1-x_3)])\Big\}H(x_3-x_1-x_2),
\end{aligned}
\tag{B 1}
$$

$$
\begin{aligned}
&(D_+^{(4)} - D_-^{(4)})_{01}\\
&= -\frac{\mathrm{i}^{1-2\nu}(1+2\nu)^2}{16\nu\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3\,B(\tilde{X}+x_1-x_3)\\
&\quad\times B(\tilde{X}+x_2-x_3)\overline{B}(\tilde{X}+x_1+x_2-2x_3)(x_1x_2x_3)^{-3/2-\nu}(x_3-x_1-x_2)^{1/2-\nu}\\
&\quad\times\Big\{-\frac{\mathrm{i}}{2\nu}[r_{10}^{(3a)}-(1+6\nu)\Delta_3]x_3[x_1(x_1-x_3)^2+x_2(x_2-x_3)^2]\\
&\quad+\frac{\overline{u}_0'}{\overline{q}_0'}x_1x_2(x_1+x_2-2x_3)-\mathrm{i}[\Delta_1+2(1+2\nu)\Delta_3]x_1x_2x_3(2x_3-x_1-x_2)\\
&\quad+\frac{[(4\nu^2-4\nu-1)\overline{u}_0'-(1+\nu)\overline{u}_0 r_{11}^{(3a)}]}{(1+2\nu)\nu\overline{q}_0'}x_3[x_1(x_3-x_1)+x_2(x_3-x_2)]\Big\}\\
&\quad\times H(x_3-x_1-x_2),
\end{aligned}
\tag{B 2}
$$

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$$(D_+^{(4)} - D_-^{(4)})_{20}$$

$$= \frac{\mathrm{i}^{-2\nu}(1+2\nu)^2\overline{u}_0'}{32\nu\overline{u}_0^2\overline{q}_0'\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}$$
$$\times\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3\,(x_1x_2x_3)^{-3/2-\nu}$$
$$\times(x_1-x_2)^2(x_1+x_2)^{1/2+\nu}$$
$$\times(2x+x_1+x_2)^{-3/2-\nu}(x_3-2x-x_1-x_2)^{1/2-\nu}$$
$$\times\Bigg\{B(\tilde{X}+x+x_1-x_3)B(\tilde{X}+x+x_2-x_3)\overline{B}_{\tilde{X}}(\tilde{X}+2x+x_1+x_2-2x_3)$$
$$\times\Bigg[2(2x+x_1+x_2)^2-\left(\frac{(1+6\nu)\overline{u}_0'-\overline{u}_0r_{11}^{(3a)}}{2\nu\overline{u}_0'}\right)x_3(2x+x_1+x_2)$$
$$+\left(\frac{(1-6\nu)\overline{u}_0'-\overline{u}_0r_{11}^{(3a)}}{\nu\overline{u}_0'}\right)x_3^2\Bigg]$$
$$+(B(\tilde{X}+x+x_1-x_3)B(\tilde{X}+x+x_2-x_3))_{\tilde{X}}$$
$$\times\overline{B}(\tilde{X}+2x+x_1+x_2-2x_3)$$
$$\times(2x+x_1+x_2+2x_3)(2x+x_1+x_2-2x_3)\Bigg\}$$
$$\times H(x_3-2x-x_1-x_2), \tag{B 3}$$

$$(D_+^{(4)} - D_-^{(4)})_{21}$$

$$= -\frac{\mathrm{i}^{1-2\nu}(1+2\nu)^3}{16\nu\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}$$
$$\times\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3\,B(\tilde{X}+x+x_1-x_3)$$
$$\times B(\tilde{X}+x+x_2-x_3)\overline{B}(\tilde{X}+2x+x_1+x_2-2x_3)$$
$$\times(x_1x_2x_3)^{-3/2-\nu}(x_1-x_2)^2(x_1+x_2)^{1/2+\nu}$$
$$\times(2x+x_1+x_2)^{-3/2-\nu}(x_3-2x-x_1-x_2)^{1/2-\nu}$$
$$\times\Bigg\{(\Delta_3-\tfrac{1}{2}\Delta_1)(2x+x_1+x_2)x_3+\Delta_1x_3^2$$
$$-\frac{\mathrm{i}}{2\overline{q}_0'}\left(\overline{u}_0'-\frac{\overline{u}_0r_{11}^{(3a)}}{(1+2\nu)}\right)(2x+x_1+x_2)$$
$$+\frac{\mathrm{i}}{\overline{q}_0'}\left(\overline{u}_0'-\frac{2\overline{u}_0r_{11}^{(3a)}}{(1+2\nu)}\right)x_3\Bigg\}$$
$$\times H(x_3-2x-x_1-x_2), \tag{B 4}$$

$$
\begin{aligned}
&(D_+^{(4)} - D_-^{(4)})_{22} \\
&\quad = \frac{\mathrm{i}^{1-2\nu}(1+2\nu)^3}{32\nu\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu} \\
&\qquad \times \int_0^\infty \mathrm{d}\overline{x} \int_0^\infty \mathrm{d}\hat{x} \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\,(x_1x_2x_3)^{-3/2-\nu} \\
&\qquad \times (x_3 - 2\overline{x} - 2\hat{x} - x_1 - x_2)^{1/2-\nu}(x_1-x_2)^2(x_1+x_2)^{1/2+\nu} \\
&\qquad \times (2\hat{x}+x_1+x_2)^{-1-2\nu}(2\overline{x}+2\hat{x}+x_1+x_2)^{-1/2+\nu} \\
&\qquad \times (2x_3 - 2\overline{x} - 2\hat{x} - x_1 - x_2)\overline{B}(\tilde{X}+2\overline{x}+2\hat{x}+x_1+x_2-2x_3) \\
&\qquad \times [-\mathrm{i}(\overline{u}_0/\overline{q}_0')(B(\tilde{X}+\overline{x}+\hat{x}+x_1-x_3)B(\tilde{X}+\overline{x}+\hat{x}+x_2-x_3))_{\tilde{X}} \\
&\qquad + 2r_{10}^{(3a)}B(\tilde{X}+\overline{x}+\hat{x}+x_1-x_3)B(\tilde{X}+\overline{x}+\hat{x}+x_2-x_3)] \\
&\qquad \times H(x_3 - 2\overline{x} - 2\hat{x} - x_1 - x_2) \\
&\qquad + \frac{\mathrm{i}^{1-2\nu}(1+2\nu)^3}{32\nu\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu} \\
&\qquad \times \int_0^\infty \mathrm{d}\overline{x} \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\,(x_1x_2x_3)^{-3/2-\nu} \\
&\qquad \times (x_3 - 2\overline{x} - x_1 - x_2)^{1/2-\nu}(x_1+x_2)^{1/2-\nu} \\
&\qquad \times (2\overline{x}+x_1+x_2)^{-1/2+\nu}(2x_3 - 2\overline{x} - x_1 - x_2)\overline{B}(\tilde{X}+2\overline{x}+x_1+x_2-2x_3) \\
&\qquad \times \Bigg[\Bigg\{-(\Delta_1+\Delta_2)(x_1^2+x_2^2) + 2(\Delta_1+\Delta_2+2\Delta_3)x_1x_2 \\
&\qquad - \frac{\mathrm{i}}{\overline{q}_0'}\left(\overline{u}_0' + \frac{\overline{u}_0 r_{11}^{(3a)}}{(1+2\nu)}\right)(x_1+x_2)\Bigg\} \\
&\qquad \times B(\tilde{X}+\overline{x}+x_1-x_3)B(\tilde{X}+\overline{x}+x_2-x_3) \\
&\qquad - \frac{\mathrm{i}}{(1+2\nu)\overline{q}_0'}\Bigg\{\frac{1}{2\nu}[(1+6\nu)\overline{u}_0' - \overline{u}_0 r_{11}^{(3a)}]\,x_1x_2 \\
&\qquad \times (B(\tilde{X}+\overline{x}+x_1-x_3)B(\tilde{X}+\overline{x}+x_2-x_3))_{\tilde{X}} \\
&\qquad - \overline{u}_0' x_2^2 B(\tilde{X}+\overline{x}+x_1-x_3)B_{\tilde{X}}(\tilde{X}+\overline{x}+x_2-x_3) \\
&\qquad - \overline{u}_0' x_1^2 B_{\tilde{X}}(\tilde{X}+\overline{x}+x_1-x_3)B(\tilde{X}+\overline{x}+x_2-x_3) \\
&\qquad - \frac{1}{2\nu}(\overline{u}_0' - \overline{u}_0 r_{11}^{(3a)})(x_2^2 B_{\tilde{X}}(\tilde{X}+\overline{x}+x_1-x_3)B(\tilde{X}+\overline{x}+x_2-x_3) \\
&\qquad + x_1^2 B(\tilde{X}+\overline{x}+x_1-x_3)B_{\tilde{X}}(\tilde{X}+\overline{x}+x_2-x_3))\Bigg\}\Bigg] \\
&\qquad \times H(x_3 - 2\overline{x} - x_1 - x_2), \qquad \text{(B 5)}
\end{aligned}
$$

where $r_{10}^{(3a)}$, $r_{11}^{(3a)}$ are defined by (4.22a,b), and Δ_1, Δ_2, Δ_3 are defined in Appendix A (see (A 1a–c)).

Appendix C. The kernel functions

For our numerical calculations it is necessary to convert the jump expression into kernel form, (6.2). The kernels in equation (6.2) can be written

$$
\begin{aligned}
G_1(\sigma) = &-\frac{\mathrm{i}}{4\Gamma(1-2\nu)}\sigma^{1-2\nu}(1+\sigma)^{-3/2-\nu}\left[\frac{(1-2\nu)}{2\nu(1+2\nu)}[r_{10}^{(3a)}-(1+6\nu)\Delta_3](1+\sigma)\right.\\
&\quad\times\left[2\sigma F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\nu,1-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right.\\
&\quad\left.+F_1\left(\frac{3}{2}-\nu,\frac{3}{2}+\nu,\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right]\\
&\quad\left.-[\Delta_1+2(1+2\nu)\Delta_3]F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right]\\
&+\frac{1}{\Gamma^2(\frac{1}{2}-\nu)}[\sigma(1-\sigma)]^{1-2\nu}\\
&\quad\times\int_0^1 \mathrm{d}r\,\frac{r^{-1/2-\nu}(1-r)^{1/2-\nu}(1-\sigma r)^{-3/2+3\nu}(1+\sigma r)^{-1/2-\nu}}{(1-\sigma^2 r)^{3/2+\nu}}\\
&\quad\times[(1-\sigma)(1+\sigma r)\Delta_1-(1-\sigma^2 r)(\tfrac{1}{2}\Delta_1-\Delta_3)]\\
&\quad\times F_1(-\tfrac{1}{2}-\nu,-\tfrac{1}{4}-\tfrac{1}{2}\nu;\tfrac{3}{4}-\tfrac{1}{2}\nu;\sigma^2 r^2)\\
&-\frac{(1+2\nu)}{2\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1+\sigma)(1-\sigma)^{1/2-3\nu}\\
&\quad\times\int_0^1 \mathrm{d}r\,\frac{r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{-3/2-\nu}}{(1+\sigma-\sigma r)^2}\left[\frac{1}{(1+2\nu)}(\Delta_1+\Delta_2)(1-\sigma r)^2\right.\\
&\quad\times F_1\left(-\frac{1}{2}-\nu,\frac{3}{2}-3\nu,-\frac{1}{2}+\nu,\frac{1}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\\
&\quad+\frac{1}{(3-2\nu)}(\Delta_1+\Delta_2)\sigma^2(1-r)^2\\
&\quad\times F_1\left(\frac{3}{2}-\nu,\frac{3}{2}-3\nu,-\frac{1}{2}+\nu,\frac{5}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\\
&\quad-\frac{2(\Delta_1+\Delta_2+2\Delta_3)}{(1-2\nu)}\sigma(1-r)(1-\sigma r)\\
&\quad\left.\times F_1\left(\frac{1}{2}-\nu,\frac{3}{2}-3\nu,-\frac{1}{2}+\nu,\frac{3}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\right]\\
&-\frac{(1+2\nu)r_{10}^{(3a)}}{2\Gamma^2(\frac{1}{2}-\nu)}\sigma^{2-2\nu}(1-2\sigma)(1-\sigma)^2(1+\sigma)\\
&\quad\times\int_0^1 \mathrm{d}r\,r^{1/2-\nu}(1-r)^{1/2-\nu}(1+\sigma-\sigma r)^{-3/2-\nu}(1+\sigma-2\sigma r)^{-1/2+\nu}\\
&\quad\times\int_0^1 \mathrm{d}v\,[1+\sigma-2\sigma r-2\sigma(1-r)v]^{-1-2\nu}\int_0^1 \mathrm{d}u\,[1-(u+v)]^{-3/2-\nu}\\
&\quad\times[1-\sigma r-(1-r)(u+v)]^{-3/2-\nu}[1+\sigma-2\sigma r-2\sigma(1-r)(u+v)]^{1/2+\nu},
\end{aligned}\tag{C 1}
$$

$$
\begin{aligned}
G_2(\sigma) = {} & \frac{1}{4\overline{q}_0'\Gamma(1-2\nu)}\frac{\sigma^{1-2\nu}}{(1+\sigma)^{3/2+\nu}} \\
& \times\left\{\overline{u}_0'(1+\sigma)F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{3}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right. \\
& +\frac{[(4\nu^2-4\nu-1)\overline{u}_0'+(1+\nu)\overline{u}_0 r_{11}^{(3a)}]}{\nu(1+2\nu)} \\
& \times\sigma\left[F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{3}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right. \\
& \left.\left.+\frac{(1-2\nu)}{4(1-\nu)}F_1\left(\frac{3}{2}-\nu,\frac{3}{2}+\nu,\frac{3}{2}+\nu,3-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right]\right\} \\
& -\frac{\mathrm{i}}{2(1+2\nu)\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\frac{\sigma^{1-2\nu}}{(1-\sigma)^{2\nu}} \\
& \times\int_0^1 \mathrm{d}r\, r^{-1/2-\nu}(1-r)^{1/2-\nu}(1-\sigma r)^{-1/2+3\nu} \\
& \times(1+\sigma r)^{-3/2-\nu}(1-\sigma^2 r)^{-3/2-\nu} \\
& \times\{(1-\sigma)(1+\sigma r)[(1+2\nu)\overline{u}_0'-\overline{u}_0 r_{11}^{(3a)}] \\
& -2(1-\sigma^2 r)[(1+2\nu)\overline{u}_0'-2\overline{u}_0 r_{11}^{(3a)}]\} \\
& \times F(-\tfrac{1}{2}-\nu,-\tfrac{1}{4}-\tfrac{1}{2}\nu;\tfrac{3}{4}-\tfrac{1}{2}\nu;\sigma^2 r^2) \\
& -\frac{\mathrm{i}[\overline{u}_0 r_{11}^{(3a)}+(1+2\nu)\overline{u}_0']}{2\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1+\sigma)(1-\sigma)^{1/2-3\nu} \\
& \times\int_0^1 \mathrm{d}r\, r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{-1/2+\nu}(1+\sigma-\sigma r)^{-2} \\
& \times\left\{\frac{1}{(1-2\nu)}\sigma(1-r)(1-\sigma)^{-1}\right. \\
& \times F_1\left(\frac{1}{2}-\nu,\frac{1}{2}-3\nu,-\frac{1}{2}+\nu,\frac{3}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right) \\
& -\frac{1}{(1+2\nu)}(1-\sigma r) \\
& \left.\times F_1\left(-\frac{1}{2}-\nu,\frac{1}{2}-3\nu,-\frac{1}{2}+\nu,\frac{1}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\right\},
\end{aligned}
\qquad (C\,2)
$$

$$
\begin{aligned}
G_3(\sigma) = {} & \frac{\mathrm{i}}{4\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1+\sigma)(1-\sigma)^{1/2-3\nu} \\
& \times\int_0^1 \mathrm{d}r\,\frac{r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{-1/2+\nu}}{(1+\sigma-\sigma r)^2} \\
& \times\left[\frac{[\overline{u}_0 r_{11}^{(3a)}-(1+6\nu)\overline{u}_0']}{\nu(1-2\nu)}\sigma(1-r)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times F_1\left(\frac{1}{2}-\nu,\frac{3}{2}-3\nu,-\frac{1}{2}+\nu,\frac{3}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\\
&+\frac{2\overline{u}_0'}{(3-2\nu)}\frac{\sigma^2(1-r)^2}{1-\sigma r}\\
&\times F_1\left(\frac{3}{2}-\nu,\frac{3}{2}-3\nu,-\frac{1}{2}+\nu,\frac{5}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\\
&-\frac{[\overline{u}_0'-r_{11}^{(3a)}\overline{u}_0]}{\nu(1+2\nu)}(1-\sigma r)\\
&\times F_1\left(-\frac{1}{2}-\nu,\frac{3}{2}-3\nu,-\frac{1}{2}+\nu,\frac{1}{2}-\nu;\frac{\sigma(1-r)}{1-\sigma r};-\frac{\sigma(1-r)}{1-\sigma r}\right)\Bigg]\\
-&\frac{\mathrm{i}\overline{u}_0'}{2(1+2\nu)\overline{q}_0'\Gamma(1-2\nu)}\sigma^{1-2\nu}(1+\sigma)^{-1/2-\nu}\\
&\times\left\{F_1\left(\frac{3}{2}-\nu,\frac{3}{2}+\nu,\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right.\\
&-\frac{[(1+6\nu)\overline{u}_0'-r_{11}^{(3a)}\overline{u}_0]}{4\nu\overline{u}_0'}F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\\
&+2\sigma\left[\frac{(1-2\nu)}{(1+2\nu)}F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\nu,1-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right.\\
&\left.\left.+F_1\left(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{1+\sigma}\right)\right]\right\}\\
-&\frac{\mathrm{i}\overline{u}_0'}{2(1+2\nu)\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1-\sigma)^{1/2-\nu}\\
&\times\int_0^1\mathrm{d}r\,\frac{r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{1/2+\nu}}{(1+\sigma-\sigma r)^{3/2+\nu}}\\
&\times(1+\sigma-2\sigma r)^{-3/2-\nu}(1+\sigma)(3+3\sigma-4\sigma r)\\
&\times F\left(-\frac{1}{2}-\nu,-\frac{1}{4}-\frac{1}{2}\nu;\frac{3}{4}-\frac{1}{2}\nu;\frac{\sigma^2(1-r)^2}{(1-\sigma r)^2}\right)\\
+&\frac{\mathrm{i}(1+2\nu)\overline{u}_0}{4\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{2(1-\nu)}(1-2\sigma)(1-\sigma)^2(1+\sigma)\\
&\times\int_0^1\mathrm{d}r\,r^{1/2-\nu}(1-r)^{1/2-\nu}(1+\sigma-\sigma r)^{-3/2-\nu}(1+\sigma-2\sigma r)^{-1/2+\nu}\\
&\times\int_0^1\mathrm{d}v[1+\sigma-2\sigma r-2\sigma(1-r)v]^{-1-2\nu}\int_0^1\mathrm{d}u(1-u-v)^{-3/2-\nu}\\
&\times[1-\sigma r-(1-r)(u+v)]^{-3/2-\nu}[1+\sigma-2\sigma r-2\sigma(1-r)(u+v)]^{1/2+\nu},
\end{aligned}
\tag{C 3}
$$

$$
\begin{aligned}
G_4(\sigma) = {} & \frac{\mathrm{i}}{4\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1+\sigma)(1-\sigma)^{1/2-3\nu} \\
& \times \int_0^1 \mathrm{d}r\, \frac{r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{-1/2+\nu}}{(1+\sigma-\sigma r)^2} \\
& \times \Bigg[\frac{[\overline{u}_0 r_{11}^{(3a)} - (1+6\nu)\overline{u}_0']}{\nu(1-2\nu)}\sigma(1-r) \\
& \times F_1\left(\frac{1}{2}-\nu, \frac{3}{2}-3\nu, -\frac{1}{2}+\nu, \frac{3}{2}-\nu; \frac{\sigma(1-r)}{1-\sigma r}; -\frac{\sigma(1-r)}{1-\sigma r}\right) \\
& + \frac{[\overline{u}_0' - r_{11}^{(3a)}\overline{u}_0]}{\nu(3-2\nu)}\frac{\sigma^2(1-r)^2}{1-\sigma r} \\
& \times F_1\left(\frac{3}{2}-\nu, \frac{3}{2}-3\nu, -\frac{1}{2}+\nu, \frac{5}{2}-\nu; \frac{\sigma(1-r)}{1-\sigma r}; -\frac{\sigma(1-r)}{1-\sigma r}\right) \\
& - \frac{2\overline{u}_0'}{(1+2\nu)}(1-\sigma r) \\
& \times F_1\left(-\frac{1}{2}-\nu, \frac{3}{2}-3\nu, -\frac{1}{2}+\nu, \frac{1}{2}-\nu; \frac{\sigma(1-r)}{1-\sigma r}; -\frac{\sigma(1-r)}{1-\sigma r}\right)\Bigg] \\
& - \frac{\mathrm{i}\overline{u}_0'}{2(1+2\nu)\overline{q}_0'\Gamma(1-2\nu)}\sigma^{1-2\nu}(1+\sigma)^{-1/2-\nu} \\
& \times \Bigg\{ -F_1\left(\frac{3}{2}-\nu, \frac{3}{2}+\nu, \frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right) \\
& - \frac{[(1+6\nu)\overline{u}_0' - r_{11}^{(3a)}\overline{u}_0]}{4\nu\overline{u}_0'}\sigma F_1\left(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right) \\
& + 2\Bigg[\frac{(1-2\nu)}{(1+2\nu)}\sigma F_1\left(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}+\nu, 1-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right) \\
& + F_1\left(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right)\Bigg]\Bigg\} \\
& - \frac{\mathrm{i}\overline{u}_0'}{2(1+2\nu)\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1-\sigma)^{1/2-\nu} \\
& \times \int_0^1 \mathrm{d}r\, \frac{r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{1/2+\nu}}{(1+\sigma-\sigma r)^{3/2+\nu}} \\
& \times (1+\sigma-2\sigma r)^{-3/2-\nu}(1+\sigma)(3+3\sigma-4\sigma r) \\
& \times F\left(-\frac{1}{2}-\nu, -\frac{1}{4}-\frac{1}{2}\nu; \frac{3}{4}-\frac{1}{2}\nu; \frac{\sigma^2(1-r)^2}{(1-\sigma r)^2}\right) \\
& + \frac{\mathrm{i}(1+2\nu)\overline{u}_0}{4\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{2(1-\nu)}(1-2\sigma)(1-\sigma)^2(1+\sigma) \\
& \times \int_0^1 \mathrm{d}r\, r^{1/2-\nu}(1-r)^{1/2-\nu}(1+\sigma-\sigma r)^{-3/2-\nu}(1+\sigma-2\sigma r)^{-1/2+\nu} \\
& \times \int_0^1 \mathrm{d}v[1+\sigma-2\sigma r-2\sigma(1-r)v]^{-1-2\nu}\int_0^1 \mathrm{d}u(1-u-v)^{-3/2-\nu} \\
& \times [1-\sigma r-(1-r)(u+v)]^{-3/2-\nu}[1+\sigma-2\sigma r-2\sigma(1-r)(u+v)]^{1/2+\nu}, \quad (\mathrm{C}\,4)
\end{aligned}
$$

$$G_5(\sigma) = \frac{\mathrm{i}\overline{u}_0'}{(1+2\nu)\overline{q}_0'\Gamma(1-2\nu)}\frac{\sigma^{1-2\nu}}{(1+\sigma)^{1/2+\nu}}$$
$$\times\left[\frac{1}{2}F_1\left(\frac{3}{2}-\nu, \frac{3}{2}+\nu, \frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right)\right.$$
$$-\frac{(1-2\nu)}{(1+2\nu)}\sigma F_1\left(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}+\nu, 1-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right)$$
$$\left.-(1+\sigma)F_1\left(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{1+\sigma}\right)\right]$$
$$+\frac{\mathrm{i}}{4\nu(1+2\nu)\overline{q}_0'\Gamma^2(\frac{1}{2}-\nu)}\sigma^{1-2\nu}(1-\sigma)^{1/2-\nu}$$
$$\times\int_0^1 \mathrm{d}r\, r^{1/2-\nu}(1-r)^{-1/2-\nu}(1-\sigma r)^{1/2+\nu}$$
$$\times(1+\sigma-\sigma r)^{-3/2-\nu}(1+\sigma-2\sigma r)^{-3/2-\nu}$$
$$\times F\left(-\frac{1}{2}-\nu, -\frac{1}{4}-\frac{1}{2}\nu; \frac{3}{4}-\frac{1}{2}\nu; \frac{\sigma^2(1-r)^2}{(1-\sigma r)^2}\right)\left[4\nu\overline{u}_0'(1+\sigma-2\sigma r)^2\right.$$
$$-[(1+6\nu)\overline{u}_0' - r_{11}^{(3a)}\overline{u}_0](1+\sigma-\sigma r)(1+\sigma-2\sigma r)$$
$$\left.+2[(1-6\nu)\overline{u}_0' - r_{11}^{(3a)}\overline{u}_0](1+\sigma-\sigma r)^2\right], \qquad \text{(C 5)}$$

where F is the hypergeometric function of one variable and F_1 the hypergeometric function of two variables (see Erdélyi (1953) and Abramowitz & Stegun (1964) for details). The quantities $r_{10}^{(3a)}$, $r_{11}^{(3a)}$ are defined by (4.22a, b), and Δ_1, Δ_2, Δ_3 are defined in Appendix A (see (A 1a–c)).

Appendix D. Finite-distance break-up solutions of equation (6.3)

Often, nonlinear evolution equations arising from weakly nonlinear studies permit so-called finite-distance break-up solutions (for certain ranges of parameter values). Here, we investigate such a possibility for our evolution equation (6.3).

Let us suppose that a solution to (6.3) terminates, as $\tilde{X} \to \tilde{X}_0$, as an algebraic singularity of the form

$$B(\tilde{X}) = B_* \times (\tilde{X}_0 - \tilde{X})^{-r}(1 + O(\tilde{X}_0 - \tilde{X})); \quad \mathrm{Re}[r] > 0, \qquad \text{(D 1)}$$

where the (complex) constants B_* and r are to be determined.

Substituting (D 1) into the evolution equation (6.3) leads, after some manipulation, to the conditions

$$\mathrm{Re}[r] = (3-4\nu)/2, \qquad |B_*|^2 = \frac{r}{\gamma_2(\tilde{I}_2 + r\tilde{I}_3 + r\tilde{I}_4 + \overline{r}\tilde{I}_5)}, \qquad \text{(D 2)}$$

where $\overline{r}$ denotes the complex conjugate of r, and

$$\tilde{I}_2 = \int_0^1 \mathrm{d}\sigma\, G_2(\sigma)(1+\sigma)^{-2+4\nu}\int_0^1 \mathrm{d}x\, x^{1-4\nu}(1-x)^r\left(1-\frac{\sigma x}{1+\sigma}\right)^{-r}\left(1-\frac{x}{1+\sigma}\right)^{-r}, \qquad \text{(D 3}a\text{)}$$

$$\tilde{I}_3 = \int_0^1 \mathrm{d}\sigma\, G_3(\sigma)(1+\sigma)^{-3+4\nu} \int_0^1 \mathrm{d}x\, x^{2-4\nu}(1-x)^r \left(1-\frac{\sigma x}{1+\sigma}\right)^{-r-1} \left(1-\frac{x}{1+\sigma}\right)^{-r}, \tag{D 3 b}$$

$$\tilde{I}_4 = \int_0^1 \mathrm{d}\sigma\, G_4(\sigma)(1+\sigma)^{-3+4\nu} \int_0^1 \mathrm{d}x\, x^{2-4\nu}(1-x)^r \left(1-\frac{\sigma x}{1+\sigma}\right)^{-r} \left(1-\frac{x}{1+\sigma}\right)^{-r-1}, \tag{D 3 c}$$

$$\tilde{I}_5 = \int_0^1 \mathrm{d}\sigma\, G_5(\sigma)(1+\sigma)^{-3+4\nu} \int_0^1 \mathrm{d}x\, x^{2-4\nu}(1-x)^r \left(1-\frac{\sigma x}{1+\sigma}\right)^{-r} \left(1-\frac{x}{1+\sigma}\right)^{-r}. \tag{D 3 d}$$

Note that since $\nu < -1/2$ for the modes considered, equation (D 2 *a*) shows that the condition $\mathrm{Re}[r] > 0$ is always satisfied. Taking real and imaginary parts of equation (D 2 *b*) should then enable us to determine the remaining two unknowns, $\mathrm{Im}[r]$ and $|B_*|^2$. Then the fact that $|B_*|^2$ is positive would lead to a condition on γ_2 that would determine whether an algebraic singularity occurs. However, it is impossible to solve (D 2 *b*) analytically (to determine $\mathrm{Im}[r]$ and $|B_*|^2$) and hence we can not determine analytically whether an algebraic singularity occurs.

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